Number Theory Reading Group

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Contents

1 Thursday, 9/12/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

$$\mathfrak{sl}_2(\mathbb{F}) := \{g \in \mathfrak{gl}_2(\mathbb{F}) \mid \mathrm{Tr}(g) = 0\}$$

We assume $\mathrm{char}(\mathbb{F}) = 0$ and $\mathbb{F}$ is algebraically closed.

Theorem 1.1. $\mathfrak{sl}_2(\mathbb{F})$ is semisimple

Proof. Direct computation of the Killing Form. □

Recall: if $\mathfrak{L}$ is semisimple and $\phi : \mathfrak{L} \to \mathfrak{gl}(V)$ is a representation.
$\mathfrak{L} \ni x = s + n$ abstract jordan decomposition.
$\implies \phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in $\phi(\mathfrak{L})$.
From now on, $\mathfrak{L} = \mathfrak{sl}_2(\mathbb{F}) = \mathfrak{sl}(2, \mathbb{F})$.
(V, ϕ) is a representation.
Basis of $\mathfrak{L}$:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus we have $[h, x] = 2x, [h, y] = -2y, [x, y] = h$.
Since h is diagonal, h is semisimple.
$\implies \phi(h)$ is semisimple and thus diagonalizeable. $\in \mathrm{End}(V)$.
We can decompose $V = \bigoplus_\lambda V_\lambda$ where $V_\lambda = \{v \in V \mid hv = \lambda v\}$ for all $\lambda \in \mathbb{F}$.
We say V_λ is a <u>weight space</u> with λ as its weight.

Lemma 1.2 (7.1)**.** Suppose $v \in V_\lambda$. Then,

1) $xv \in V_{\lambda+2}$

2) $yv \in V_{\lambda-2}$

Proof. 1) $h(xv) = [h,x]v + x(hv) = 2xv + \lambda xv = (\lambda+2)xv$

2) $h(yv) = [h,y]v + y(hv) = -2yv + \lambda yv = (\lambda-2)yv$

□

$$\begin{tikzcd} V_{\lambda-2} & V_\lambda \arrow[loop above, "h"] \arrow[l, bend left, "y"] \arrow[r, bend right, "x"'] & V_{\lambda+2} \end{tikzcd}$$

Note that $\dim V < \infty$
Thus, $\exists v \in V$ such that $x \cdot v = 0$.
Such a v is called a maximal vector.
For now, assume V is irreducible.
Let v_0 be a maximal vector with weight λ.

Definition. For $i > 0$ integer, $v_i = \frac{y^i \cdot v_0}{i!}$
Also, $v_{-1} = 0$.

Lemma 1.3 (7.2)**.** 1) $h \cdot v_i = (\lambda - 2i)v_i$

2) $y \cdot v_i = (i+1)v_{i+1}$

3) $x \cdot v_i = (\lambda - i + 1)v_{i-1}$

Proof. 1) We use induction. Base case is clear.
Assume it is true for $i-1$.
$v_{i-1} \in V_{\lambda-2(i-1)}$
Thus, $v_i = \frac{1}{i} \cdot yv_{i-1}$
Lemma 7.1 implies $v_i \in V_{\lambda-2i}$.

2) $y \cdot v_i = (i+1)v_{i+1}$ by definition of v_i.

3) $ix \cdot v_i = x(yv_{i-1}) = [x,y]v_{i-1} + yxv_{i-1} = hv_{i-1} + yxv_{i-1} = (\lambda - 2(i-1))v_{i-1} + (\lambda - i + 2)yv_{i-2} = i(\lambda - i + 1)v_{i-1}$

□

$\dim V < \infty$ so it must end at some point.
So, at some point, it'll become 0. $v_0, \cdots, v_m \neq 0, v_{m+1} = 0$.

Definition. m is the integer so that $v_m \neq 0, v_{m+1} = 0$.

By Lemma 7.2,
$\text{span}\{v_0, \cdots, v_m\}$ is a sub-representation of V.
Since V is irreducible,
$V = \text{span}\{v_0, \cdots, v_m\}$
Note: by 7.2(3),
$0 = x \cdot v_{m+1} = (\lambda - m)v_m$
Since $v_m \neq 0$ we have $\lambda = m$.
Thus, $\dim V = m + 1 = \lambda + 1$
Here m is the highest weight.

$$V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$$

Construction. Suppose $L \curvearrowright \mathbb{F}[X, Y]$ [as a $\mathbb{F}$-space].
$\rho(x) = X\frac{\partial}{\partial Y}$
$\rho(y) = Y\frac{\partial}{\partial X}$
$\rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$
Consider subrepresentations $\mathbb{F}[X, Y]_m$ [symmetric polynomials of degree m, dimension $m + 1$].

$$\begin{tikzcd}
0 & X^m \arrow[l, bend left, "x"] \arrow[loop above, "h"] \arrow[r, bend right, "y"'] & X^{m-1}Y & \cdots
\end{tikzcd}$$

2 Thursday, 9/19/2024, Representation of $\mathfrak{sl}_2(\mathbb{F})$ by Hechi

Root Space Decomposition

Let $\mathcal{L}$ be a non-zero semisimple lie algebra over $\mathbb{F}$ with $\operatorname{char} \mathbb{F} = 0$ and $\mathbb{F}$ algebraically closed.

Definition (Toral Subalgebra)**.** A subalgebra $\mathcal{H} \subseteq \mathcal{L}$ toral if it consists of semisimple elements.

Remark. If every element in $\mathcal{L}$ is ad-nilpotent, then by Engel's Theorem $\mathcal{L}$ is nilpotent. Thus it is not semisimple.
So, there exists a non-zero toral subalgebra.

Fix $\mathcal{H}$ to be the maximal toral subalgebra. A maximal subalgebra exists since $\mathcal{L}$ is finite dimensional.

Lemma 2.1 (8.1)**.** A toral subalgebra $\mathcal{T}$ is abelian.

Proof. Suppose $x \in \mathcal{T}$. We will prove that $\operatorname{ad}_{\mathcal{T}} x = 0$ [as a map].
$\operatorname{ad}_T x$ is diagonalizeable. Assume some eigenvalue is non-zero. Then, we can find eigenvactor $y \in T$ with eigenvalue $a \neq 0$. So, $[x, y] = ay$.
Now, $\operatorname{ad}_T y(x) = [y, x] = -ay$. Since $[y, y] = 0$ we see that $-ay$ is an eigenvector of $\operatorname{ad}_T y$ with eigenvalue 0.
$\operatorname{ad}_T y$ is also diagonalizeable. Suppose $v_1, \cdots, v_n$ is the eigenbasis of $\operatorname{ad}_T y$ with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then $x = a_1 v_1 + \cdots + a_n v_n$ for $a_i \in \mathbb{F}$.
WLOG, $v_1 = y$.

$$[y, x] = a_1\lambda_1 v_1 + \cdots + a_n \lambda_n v_n = -ay$$

By comparing coefficients, $a_1\lambda_1 = -a$. But $\lambda_1 = 0$. This is a contradiction.

□

Now, we fix $\mathcal{H}$ to be a maximal toral subalgebra. It is not necessarily unique.
Note that $\operatorname{ad} H$ is a commuting family in $\operatorname{End}(\mathcal{L})$. From linear algebra we know that $\operatorname{ad} H$ is simultaneously diagonalizeable.

Definition (Root Space Decomposition)**.** Suppose $\mathcal{H}^*$ is the dual space of $\mathcal{H}$. We can write:

$$\mathcal{L} = \bigoplus_{\alpha \in H^*} \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \forall h \in H\}$$

$$= \mathcal{L}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathcal{L}\alpha$$

where $\Phi = \{\alpha \in H^* \setminus \{0\} \mid \mathcal{L}\alpha \neq 0\}$ and $\mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$ [the centralizer].
This is called the root space decomposition.

Example. $\mathfrak{sl}_2(\mathbb{F})$ has basis:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then the root space decomposition is:

$$\mathfrak{sl}_2(\mathbb{F}) = \mathcal{H} \oplus \mathcal{L}_{-2} \oplus \mathcal{L}_2$$

$\mathcal{L}_{-2}$ contains the linear form sending h to -2.

Proposition 2.2 (8.1)**.** Let $\alpha, \beta \in \mathcal{H}^*$. Then,

1) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ [by Jacobi Identity]

2) $\alpha \neq 0 \implies \forall x \in L_\alpha$ is nilpotent [by 1]

3) $\alpha + \beta \neq 0 \implies L_\alpha \perp L_\beta$ w.r.t. the Killing Form.

Proof of 3. Find $h \in \mathcal{H}$ such that $(\alpha+\beta)(h) \neq 0$. Then,

$$\kappa([h,x],y) = -\kappa([x,h],y) = -\kappa(x,[h,y])$$

$$\implies (\alpha+\beta)(h)\kappa(x,y) = 0$$

□

In particular, $L_0 \perp L_\alpha$ when $\alpha \in \Phi$.

Corollary 2.3 (8.1)**.** The Killing Form restricted to $\mathcal{L}_0$, $\kappa|_{\mathcal{L}_0}$ is non-degenerate.

Proposition 2.4 (8.2)**.** $\mathcal{H} = \mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$.

Proof. Tedious linear algebra □

Corollary 2.5 (8.2)**.** The Killing Form restricted to $\mathcal{H}$, $\kappa|_{\mathcal{H}}$ is non-degenerate.

This implies, the map $H \to H^*$ given by $x \mapsto \kappa(x,-)$ is an isomorphism.
For each $\phi \in \mathcal{H}^*$ we can define $t_\phi \in \mathcal{H}$ to be the pre-image of this isomorphism. So it satisfies

$$\phi(h) = \kappa(t_\phi, h) \quad \forall h \in \mathcal{H}$$

Proposition 2.6 (8.3)**.** 1) Φ spans $\mathcal{H}^*$

2) If $\alpha \in \Phi$ then $-\alpha \in \Phi$

3) $x \in \mathcal{L}_\alpha, y \in \mathcal{L}_{-\alpha} \implies [x,y] = \kappa(x,y)t_\alpha$

4) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$

5) $\dim[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = 1$, spanned by t_α

6) Pick any non-zero $x_\alpha \in L_\alpha \setminus \{0\}$. Then there exists $y_\alpha \in \mathcal{L}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha \coloneqq [x_\alpha, y_\alpha]$ spans a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{F})$, with the isomorphism $x_\alpha \mapsto x, y_\alpha \mapsto y, h_\alpha \mapsto h$

7) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$.

If V is a $\mathfrak{sl}_2(\mathbb{F})$-module, recalling that $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda \text{ eigenspaces of } h$$

Recall that all $\mathfrak{sl}_2(\mathbb{F})$-module is of the form:

$$\mathfrak{sl}_2(\mathbb{F}) \curvearrowright \mathbb{F}[X,Y]$$

$$\rho(x) = X\frac{\mathrm{d}}{\mathrm{d}Y}, \rho(y) = Y\frac{\mathrm{d}}{\mathrm{d}X}, \rho(h) = X\frac{\mathrm{d}}{\mathrm{d}X} - Y\frac{\mathrm{d}}{\mathrm{d}Y}$$

and $V = \mathbb{F}[X,Y]_m$ [homogeneous polynomials of degree m] is irreducible and give us all irreducible representations.
Then we have:

$$V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$$

Where V_m is generated by X^m and V_{-m} is generated by Y^m

$$\begin{tikzcd}
Y^m \arrow[r, bend right, "x"'] & XY^{m-1} \arrow[l, bend right, "y"'] \arrow[loop above, "h"] & \cdots & X^m
\end{tikzcd}$$

If m even, $0 \neq V_0 \subseteq V$
If m odd, $0 \neq V_1 \subseteq V$

Corollary 2.7. V is a $\mathfrak{sl}_2(\mathbb{F})$-module. Then $\dim V_0 + \dim V$ gives the number of summands in the irreducible decomposition of V.

Consider $\mathcal{S}_\alpha = \text{span}\{x_\alpha, y_\alpha, h_\alpha\} \cong \mathfrak{sl}_2(\mathbb{F})$ and its adjoint representation ($\mathcal{L}$ is an $\mathcal{S}_\alpha$ module).
Fix $\alpha \in \Phi$ and let $\mathcal{M} = \mathcal{H} + \sum_{c \in \mathbb{F}^\times} \mathcal{L}_{c\alpha}$.
By proposition 8.1, $\mathcal{M}$ is a submodule of $\mathcal{L}$ [since $[\mathcal{L}_{c_1\alpha}, \mathcal{L}_{c_2\alpha}] \subseteq \mathcal{L}_{(c_1+c_2)\alpha}$].
If $0 \neq x \in \mathcal{L}_{c\alpha}$ we see that $[h_\alpha, x] = c\alpha(h_\alpha) \cdot x = 2cx$
$\implies 2c \in \mathbb{Z}$ and a $\underbrace{\text{weight}}_{\text{eigenvalue}}$ of h_α is 0 or an integer multiple of $\frac{1}{2}$.

Then $\mathcal{M} = \underbrace{\ker \alpha}_{\text{vectors of weight } 0} + \underbrace{\mathbb{F} \cdot h_\alpha}_{\text{weight } 0, \pm 2}$
Therefore, $\mathcal{M}$ contains vectors of weight only 0 or ± 2.
Therefore, if $\alpha \in \Phi$ we have $c = \pm 1$.
$\mathcal{M} = \mathcal{H} + \mathcal{S}_\alpha$. Suppose h_α^c is the complement of h_α in $\mathcal{H}$.
Then, $\mathcal{H} + \mathcal{S}_\alpha = \underbrace{h_\alpha^c}_{\text{abelian}} + \underbrace{\mathcal{S}_\alpha}_{\text{irreducible}}$ has $\dim \mathcal{H} - 1 + 1 = \dim \mathcal{H} = \dim \mathcal{M} - 2$ irreducible summands.
On the other hand, the number of irreducible summands of $\mathcal{M}$ is $\underbrace{\dim \mathcal{M}_0}_{\dim \mathcal{M} - 2} + \underbrace{\dim \mathcal{M}_1}_{0}$
Therefore, $\mathcal{H} + \mathcal{S}_\alpha \subseteq \mathcal{M}$ must be equal.
Therefore, $\dim \mathcal{L}_\alpha = 1$.
Now, suppose $\beta \neq \pm\alpha \in \Phi$. Then, $\exists r, q$ such that $\beta - r\alpha, \beta - (r-1)\alpha, \cdots, \beta + q\alpha$ are roots and outside outside these, i.e. $\beta - (r+1)\alpha, \beta + (q+1)\alpha$ are not.
To see this, suppose $K = \sum_{i \in \mathbb{Z}} \mathcal{L}_{\beta + i\alpha} \subseteq \mathcal{L}$ is a $\mathcal{S}_\alpha$-submodule. We know that $\beta + i\alpha \neq 0$.
Weights:

$$\beta(h_\alpha) + i\alpha(h_\alpha) = \beta(h_\alpha) + 2i$$

So, weights are either all even or all odd.
Therefore, K is irreducible.
Consider $\gamma, \delta \in \mathcal{H}^*$.
Define $(\gamma, \delta) = \kappa(t_\gamma, t_\delta)$ on $E_\mathbb{Q} = \text{span}_\mathbb{Q}(\Phi)$ then $(\cdot, \cdot)$ extends to $E = E_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R}$ is positive definite.
Then E is an Euclidean Space.

(Φ, E) is called a root system.

3 Thursday, 9/26/2024, Root Systems by Zoia

Let E be an euclidean space. Suppose (α, β) is a symmetric bilinear form on E. Reflection in E fixes some hyperplane H. If α is perpendicular to H then the reflection sends α to $-\alpha$

Consider $\alpha \in E$ and $P_\alpha = \{\beta \in E \mid (\alpha, \beta) = 0\}$ the hyperplane perpendicular to α. Suppose σ_α is the reflection w.r.t. this hyperplane. Then,

$$\operatorname{proj}_\alpha(\beta) = \frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

$$\sigma_\alpha(\beta) = \beta - 2\operatorname{proj}_\alpha(\beta) = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

Define:

$$\langle \beta, \alpha \rangle = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

Note that $\langle \beta, \alpha \rangle$ is linear only in β. Then,

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

Lemma 3.1. Let Φ be a finite subset of E so that Φ spans E. Suppose all reflections $\sigma_\alpha(\alpha \in \Phi)$ leaves Φ invariant. If $\sigma \in \mathrm{GL}(E)$ fixes hyperplane P of E and sends $0 \neq \alpha \in \Phi$ to $-\alpha$, then $\sigma = \sigma_\alpha$ and $P = P_\alpha$.

Proof. Suppose $\tau = \sigma\sigma_\alpha = \sigma\sigma_\alpha^{-1}$.

Then, $\tau(\Phi) = \Phi, \tau(\alpha) = \alpha$ and τ acts as id on $\mathbb{R} \cdot \alpha$ and $E/R \cdot \alpha$ eigenvalues are 1.

So we have $(T-1)^L$ where $L = \dim E$.

$\beta, \tau(\beta), \ldots \tau^k(\beta)$ $\exists k$ that fixes all $\beta \in \Phi$

Φ spans E, so $\tau^k = 1$. So $T^k - 1 = 0$.

If $m(T)$ is the minimal polynomial of τ, then:

$$m(T) \mid T^k - 1$$

$$m(T) \mid (T-1)^k$$

Therefore, $m(T) = T - 1$.

Therefore, $\tau = \mathrm{id}$.

Thus $\sigma\sigma_\alpha^{-1} = \mathrm{id} \implies \sigma = \sigma_\alpha$ ∎

Definition (Root Systems)**.** A finite subset Φ of E is a root system in E if:

1R) Φ spans E, does not contain 0.

2R) If $\alpha \in \Phi$ then only multiples of α in Φ are $\pm\alpha$.

3R) If $\alpha \in \Phi$, then σ_α leaves Φ invariant. $[\forall \beta \in \Phi, \sigma_\alpha(\beta) \in \Phi]$

4R) If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$. $\left[\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\right]$

Definition (Weyl Group)**.** Let Φ be a root system in E. Denote by $\mathcal{W}$ the subgroup of $\mathrm{GL}(E)$ generaed by $\sigma_\alpha(\alpha \in \Phi)$.

3R $\implies$ $\mathcal{W}$ is a symmetry group on Φ.

Lemma 3.2. Let Φ be a root system in E with Weyl group $\mathcal{W}$. If $\sigma \in \mathrm{GL}(E)$ leaves Φ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)} \forall \alpha \in \Phi$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle \forall \alpha, \beta \in \Phi$.

Proof. $\sigma\sigma_\alpha\sigma^{-1}(\sigma(\beta)) = \sigma\sigma_\alpha(\beta) = \sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$.

$\sigma(\beta)$ runs over Φ. $\sigma\sigma_\alpha\sigma^{-1}$ fixes $\sigma(P_\alpha)$ pointwise and $\sigma(\alpha) \to -\sigma(\alpha)$.

Therefore, $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ by the lemma.

$\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha)$

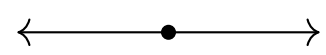
Therefore, we must have $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$. ∎

Definition (Isomorphisms)**.** Suppose Φ, Φ' are root systems with Euclidean spaces E, E'.
$(\Phi, E) \cong (\Phi', E')$ if there exists map $\varphi : E \to E'$ such that φ maps Φ to Φ' and $\forall \alpha, \beta \in \Phi$ we have $\langle \varphi(\beta), \varphi(\alpha) \rangle = \langle \beta, \alpha \rangle$.

Note that:

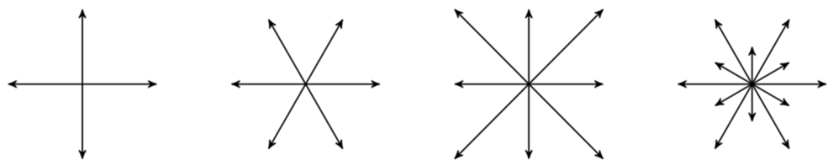
$$\sigma_{\varphi(\alpha)}(\varphi(\beta)) = \varphi(\beta) - \underbrace{\langle \varphi(\beta), \varphi(\alpha) \rangle}_{=\langle \beta, \alpha \rangle} \varphi(\alpha) = \varphi(\beta - \langle \beta, \alpha \rangle \alpha) = \varphi(\sigma_\alpha(\beta))$$

Note that, $\sigma \mapsto \varphi \sigma \varphi^{-1}$ is an isomorphism of Weyl groups.
Thus, $\mathcal{W}$ is a subgroup of $\mathrm{Aut}(\Phi)$.
Now we consider root systems of different dimensions. Suppose $L = \dim E$.
$\underline{L = 1}$: In this case, we have $\alpha, \alpha \in \Phi$ only. This gives us A_1



A_1

$\mathcal{W}(A_1) = \mathbb{Z}_2$
$\underline{L = 2}$:



$\mathrm{A}_1 \times \mathrm{A}_1$ $\quad$ A_2 $\quad$ $\mathrm{B}_2 \cong \mathrm{C}_2$ $\quad$ G_2

$\mathcal{W}(A_1 \times A_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathcal{W}(A_2) = S_3$
$\mathcal{W}(B_2) = D_4$
$\mathcal{W}(G_2) = D_6$
These are the only possible cases for $L = 2$, since:

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2\|\beta\|\|\alpha\| \cos\theta}{\|a\|\|a\|} = \frac{2\|\beta\|}{\|\alpha\|} \cos\theta \in \mathbb{Z}$$

Similarly, $\frac{2\|\alpha\|}{\|\beta\|} \cos\theta \in \mathbb{Z}$. Multiplying, $4\cos^2\theta \in \mathbb{Z} \implies 4\cos^2\theta = 0, 1, 2, 3, 4$
Thus, $\cos\theta = 0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2} \implies \theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$.

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

Table 1: Angle Root System

Lemma 3.3. Suppose α, β are non-proportional root.
If $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root.
If $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.

Proof. $\langle \alpha, \beta \rangle = 1 \implies \sigma_\beta(\alpha) = \alpha - 1\beta = \alpha - \beta \in \Phi$
If $\langle \beta, \alpha \rangle = 1$ then $\sigma_\alpha(\beta) = \beta - 1\alpha = \beta - \alpha \in \Phi$.
$\sigma_{\beta-\alpha}(\beta - \alpha) = (\beta - \alpha) - \langle \beta - \alpha, \beta - \alpha \rangle (\beta - \alpha) = \alpha - \beta \in \Phi$ ∎

4 Thursday, 10/3/2024, Simple Roots by Zoia

A root system Φ of rank l, E-Euclidean Space, $\mathcal{W}$ is the Weyl Group.

Definition. A subset Δ of Φ is called a base if:

B1) Δ is a basis of E $[|\Delta| = l]$;

B2) $\forall \beta \in \Phi, \beta = \sum_{\alpha \in \Delta} k_\alpha \cdot \alpha$, the expression is unique with k_α being integers and k_α are either all non-negative or all non-positive.

Definition. The roots from Δ are simple roots.

Definition. The height of a root β [relative to the base Δ] is:

$$\mathrm{ht}(\beta) = \sum_{\alpha \in \Delta} k_\alpha$$

Definition. We have positive roots Φ^+ and negative roots Φ^- from the sign of k_α. Furthermore $\Phi^- = -\Phi^+$.
Also, we define:

$$\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$$

Definition. $\gamma \in E$ is regular if:

$$\gamma \in E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$$

Otherwise it is called singular.
Recall that $P_\alpha = \{\beta \in E \mid (\alpha, \beta) = 0\}$

Definition. $\alpha \in \Phi^+(\gamma)$ is decomposable if $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Phi^+(\gamma)$.
α is indecomposable otherwise.

Definition. We define $\Delta(\gamma)$ to be the set of all indecomposable roots in $\Phi^+(\gamma)$.

Theorem 4.1. Any root system Φ has a base. Let $\gamma \in E$ be a regular.
Then, the set $\Delta(\gamma)$ of all the indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ.
Conversely, every base of Φ is of the form $\Delta(\gamma)$ for some γ.

Proof. We follow the following steps.
Step 1: Each root in $\Phi^+(\gamma)$ is a non-negative $\mathbb{Z}$-linear combination of $\Delta(\gamma)$.
Step 2: If $\alpha, \beta \in \Delta(\gamma)$ then $(\alpha, \beta) \leq 0$ unless $\alpha = \beta$.
Step 3: $\Delta(\gamma)$ is a linearly independent set.
Step 4: $\Delta(\gamma)$ is a base of Φ.
Step 5: Each base Δ of Φ has the form $\Delta(\gamma)$ for some regular $\gamma \in E$.
Proof of Step 1: Suppose otherwise. Then $\exists \alpha \in \Phi^+(\gamma)$ that cannot be expessed as a non-negative $\mathbb{Z}$ linear combination of $\Delta(\gamma)$.
We can have multiple such α's. We pick the α with the smallest (γ, α).
Note that $\alpha \notin \Delta(\gamma)$, since if $\alpha \in \Delta(\gamma)$ then $\alpha = 1 \cdot \alpha$, which violates the assumption. Thus, α can be written as sum of two elements in $\Phi^+(\gamma)$. Suppose $\alpha = \beta_1 + \beta_2$ so that $\beta_1, \beta_2 \in \Phi^+(\gamma)$. Then, $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$. Due to the minimality of (γ, α), they are both non-negative $\mathbb{Z}$-linear conbination of $\Delta(\gamma)$ which means so is α, a contradiction.
Proof of Step 2: Suppose otherwise. Then, $(\alpha, \beta) > 0$. β cannot be $-\alpha$, thus $\alpha - \beta$ is a root. Then either $\alpha - \beta$ or $\beta - \alpha$ is in $\Phi^+(\gamma)$. WLOG $\alpha - \beta \in \Phi^+(\gamma)$. Then $\alpha = \beta + (\alpha - \beta)$. Then α is decomposable, which is a contradiction since $\Delta(\gamma)$ consists of all indecomposable roots.
Proof of Step 3: Suppose $\sum_{\alpha \in \Delta(\gamma), r_\alpha \in \mathbb{R}} r_\alpha \cdot \alpha = 0$. r_α can be positive or negative. We redistribute so that both sides have positive coefficient:

$$\varepsilon := \sum_{\alpha} s_\alpha \alpha = \sum_{\beta} t_\beta \beta$$

Then,

$$0 \le (\varepsilon, \varepsilon) = \sum_{\alpha,\beta} \underbrace{s_\alpha t_\beta}_{\ge 0} \underbrace{(\alpha, \beta)}_{\le 0} \le 0$$

Thus, $\varepsilon = 0$. Now,

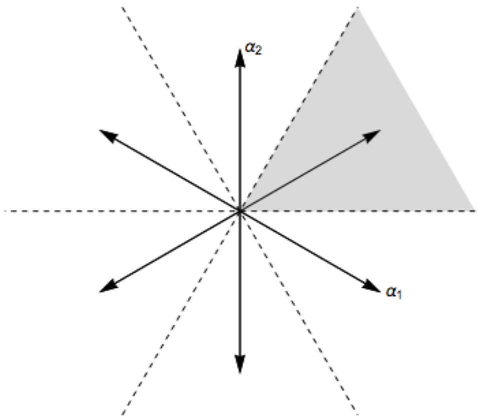
$$0 = (\gamma, \varepsilon) = \sum_{\alpha} \underbrace{s_\alpha}_{\ge 0} \underbrace{(\gamma, \alpha)}_{>0} \ge 0$$

Thus, $s_\alpha = 0$ for all $\alpha \in \Delta(\gamma)$. This implies linear independence.
Proof of Step 4: Note that $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$.
B2 is satisfied because of Step 1.
Then $\Delta(\gamma)$ spans E. Step 3 implies $\Delta(\gamma)$ is a basis of E. Thus we have B1.
Proof of Step 5: Given Δ, we select $\gamma \in E : (\alpha, \gamma) > 0 \forall \alpha \in \Delta$. B2 $\implies$ γ is regular and $\Phi^+ \subseteq \Phi^+(\gamma)$. Also, $\Phi^- \subseteq -\Phi^+(\gamma)$.
Therefore, $\Phi^+ = \Phi^+(\gamma)$. Δ consists of indecomposable elements, that is $\Delta \subseteq \Delta(\gamma)$.
Coordinates are equal, therefore $\Delta = \Delta(\gamma)$.

□

Definition (Weyl Chambers)**.** The connected components of $E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$ are called the (open) Weyl Chambers of E.
The fundamental Weyl chamber associated to γ is the open Weyl chamber containing γ. It is denoted by $\mathcal{C}(\gamma)$.
Furthermore, $C(\gamma) = C(\gamma')$ implies γ and γ' are on the same side of each hyperplane P_α. This also means $\Delta(\gamma) = \Delta(\gamma')$, so the Weyl chambers are in 1-1 correspondence with the bases.

For example: here is an open Weyl Chamber for A_2:



$\mathcal{C}(\Delta)$-fundamental Weyl chamber relative to the base $\{\alpha_1, \alpha_2\}$.

The Weyl group acts on the Weyl chambers by $\sigma(\mathcal{C}(\gamma)) = \mathcal{C}(\sigma(\gamma))$.
If $\sigma \in \mathcal{W}$ and γ is regular.
Also, $\mathcal{W}$ permutes bases. σ sends Δ to $\sigma(\Delta)$ which is another base.
Since $\sigma(\Delta(\gamma)) = \Delta(\sigma(\gamma))$ because $(\sigma\gamma, \sigma\alpha) = (\gamma, \alpha)$.

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Lemma 5.1. Let α be simple. Then σ_α permutes the positive roots other than α.

Corollary 5.2. Set $\delta = \frac{1}{2}\sum_{\beta \prec 0} \beta$. Then,

$$\sigma_\alpha(\delta) = \delta - \alpha \,\forall \alpha \in \Delta$$

Lemma 5.3. Let $\alpha_1, \cdots, \alpha_n \in \Delta$ [not necessarily distinct]. Write $\sigma_i \coloneqq \sigma_{\alpha_i}$. If $\sigma_1, \cdots, \sigma_{t-1}(\alpha_t)$ is negative, then

$$\exists s : 1 \leq s < t : \sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}$$

Corollary 5.4. If $\sigma = \sigma_1 \cdots \sigma_t$ is an exp for $\sigma \in \mathcal{W}$, t is as small as possible theen $\sigma(\alpha_t) \prec 0$.

Proof. Suppose $\sigma(\alpha_t) > 0$. Then,

$$\underbrace{\sigma_1 \cdots \sigma_t}_{t \text{ factors}} = \underbrace{\sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}}_{t-2 \text{ factors}}$$

which contradicts minimality. □

The Weyl Group

Definition. $\mathcal{W}$ is the subgroup of $GL(E)$ generated by the reflection $(\sigma_\alpha)_{\alpha \in \Phi}$.

Theorem 5.5. Let Δ be a base of Φ.

a) If $\gamma \in E$, $\underline{\gamma \text{ is regular}}$, $\exists \sigma \in \mathcal{W} : (\sigma(\gamma), \alpha) > 0 \, \forall \alpha \in \Delta$.

b) If Δ' is another base of Φ, then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathcal{W}$.

c) If α is any root $\implies \exists \sigma \in \mathcal{W} : \sigma(\alpha) \in \Delta$.

d) $\mathcal{W}$ generated by σ_α $(\alpha \in \Delta)$.

e) If $\sigma(\Delta) = \Delta, \sigma \in \mathcal{W}$ then $\sigma = \text{id}$.

Proof. We consider the subgroup $\mathcal{W}'$ generated by $\sigma_\alpha (\alpha \in \Delta)$.
For a, b, c we prove the theorem for $\mathcal{W}'$ and for d, e we prove that $\mathcal{W}' = \mathcal{W}$.
a)

$$\delta \coloneqq \frac{1}{2} \sum_{\alpha \prec 0} \alpha$$

Choose $\sigma \in \mathcal{W}'$ such that $(\sigma(\gamma), \delta)$ is as big as possible.
If α is simple then $\sigma_\alpha \sigma \in \mathcal{W}' \implies (\sigma(\gamma), \delta) \geq (\sigma_\alpha \sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha)$.
Therefore, $(\sigma(\gamma), \alpha) \geq 0$.
Furthermore, $(\sigma(\gamma), \alpha) \neq 0$ so we have strict inequality. Therefore,

$$\forall \alpha \in \Delta, (\sigma(\gamma), \alpha) > 0$$

Therefore, $\sigma(\gamma)$ is in the fundamental Weyl chamber of Δ and σ sends $\mathfrak{C}(\gamma)$ to $\mathfrak{C}(\Delta)$.
b) Since $\mathcal{W}'$ permutes the Weyl chambers by a, it also permutes the bases of Φ.
c) Hyperplanes P_β $(\beta \neq \pm\alpha)$ are distinct from hyperplane $P_\alpha \implies \exists \gamma : \gamma \in P_\alpha, \gamma \notin P_\beta$. Lets choose γ' so that γ' is close to γ such that $(\gamma', \alpha) = \varepsilon > 0$ while $|(\gamma', \beta)| > \varepsilon$ for any $\beta \neq \pm\alpha$.
Then $\alpha \in \Delta(\gamma')$.
d) We want to show that $\mathcal{W}' = \mathcal{W}$. It is enough to show that each reflection $\sigma_\alpha (\alpha \in \Phi)$ is in $\mathcal{W}'$.
Find $\sigma \in \mathcal{W}'$ such that $\beta = \sigma(\alpha) \in \Delta$ using c. Then,

$$\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma \sigma_\alpha \sigma^{-1} \implies \sigma_\alpha = \sigma^{-1} \sigma_\beta \sigma \in \mathcal{W}'$$

e) Let $\sigma(\Delta) = \Delta$ but $\sigma \neq \text{id}$. If σ is written minimally as a product of simple reflections then we have contradiction from corollary 5.4.

□

Irreducible Root System

Φ is irreducible if it cannot be partitioned intothe union of two proper subsets in the following way: each root in one set is orthogonal to each root in the other subset.
Exmaple: A_1, A_2, B_2, G_2 are irreducible. $A_1 \times A_1$ is not irreducible.
Claim: Φ is irreducible $\iff$ Δ cannot be partitioned.

Proof. $\Longleftarrow$: Suppose $\Phi = \Phi_1 \cup \Phi_2$ with $(\Phi_1, \Phi_2) = 0$.
If Δ is not wholly contained in Φ_1 or Φ_2 then it induces the partition in Δ.
Now WLOG suppose $\Delta \subset \Phi_1$. Then, $(\Delta, \Phi_2) = 0$. Since Δ spans E.
$\Longrightarrow$: Let Φ be irreducible but suppose $\Delta = \Delta_1 \cup \Delta_2$ with $(\Delta_1, \Delta_2) = 0$.
Each root is conjugate to a simple root (by theorem). Then,

$$\Phi = \Phi_1 \cup \Phi_2$$

where Φ_i is the set of roots that are conjugates with those in Δ_i.
Since $\mathcal{W}$ is generated by the σ_α where $\alpha \in \Delta$, it follows that each root in Φ_i can be obtained from obtained from Δ_i by $+$ or $-$ elements of Δ_i.
Therefore, Φ_i lies in the subspace E_i of E spanned by Δ_i.
Then, $(\Phi_1, \Phi_2) = 0$.
Since Φ is irreducible, it follows that $\Phi_1 = \varnothing$ or $\Phi_2 = \varnothing$.
Therefore, $\Delta_1 = \varnothing$ or $\Delta_2 = \varnothing$.

□

Lemma 5.6. Let Φ be irreducible. Then relative to the partial ordering $\prec$, there exists a unique maximal root β.
If $\beta = \sum_\alpha k_\alpha \alpha \, (\alpha \in \Delta)$ then all $k_\alpha > 0$.

Lemma 5.7. Let Φ be irreducible. Then $\mathcal{W}$ acts irreducibly on E. In particular, the $\mathcal{W}$-orbit of a root α spans E.

Lemma 5.8. Let Φ be irreducible. Then at most two root lengths occur in Φ and all roots of this length are conjugates under $\mathcal{W}$.

Lemma 5.9. Suppose Φ is irreducible with two distinct root lengths. Then the maximal root β of lemma 5.6 is long.

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Facts:

1) Lemma 9.2 and (R3): $\forall \alpha, \beta \in \Phi, \forall \sigma \in \mathcal{W}, \langle \beta, \alpha \rangle = \langle \sigma\beta, \sigma\alpha \rangle$.
2) Table 1 in 9.4: $\forall \alpha, \beta \in E, \alpha \neq \beta, \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2, 3$.
3) Lemma 10.1: $\forall \alpha, \beta \in \Delta, \alpha \neq \beta, \langle \alpha, \beta \rangle \leq 0$.
4) Theorem 10.3b: $\mathcal{W}$ acts transitively on the bases. 10.3c: $\forall \alpha \in \Phi, \exists \sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$. 10.3d: $\mathcal{W}$ is generated by $\sigma_\alpha (\alpha \in \Delta)$.
5) Claim 10.4: Φ: irreducible $\iff$ Δ canot be partitioned into proper $\Delta_1 \cup \Delta_2$ such that $(\Delta_1, \Delta_2) = 0$.

Classification

Fix $\Delta \subseteq \Phi$ and let $l = \dim_{\mathbb{R}} E$.

Definition. Fix an ordering $(\alpha_1, \cdots, \alpha_l)$ of Δ.
Then $(\langle \alpha_i, \alpha_j \rangle)_{ij}$ is called the cartan matrix of Φ.
The entries are called the Cartan integers.

Example. In B_2 the Cartan matrix is:

$$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

Remark. 1) The Cartan matrix depends on the chosen ordering.

2) The Cartan matrix is independent of the choice of Δ. This is because, if Δ' is another base, since σ acts transitively, there exists $\sigma \in \mathcal{W}$ such that $\sigma\Delta = \Delta'$. Using the fact $\langle \alpha_i, \alpha_j \rangle = \langle \sigma\alpha_i, \sigma\alpha_j \rangle$ we see that the matrices are the same.

3) The Cartan Matrix is nonsingular. This is because:

$$(\langle \alpha_i, \alpha_j \rangle)_{i,j} \cdot \operatorname{diag}\left(\frac{(\alpha_i, \alpha_i)}{2}, \cdots, \frac{(\alpha_l, \alpha_l)}{2} \right) = \underbrace{((\alpha_i, \alpha_j))_{ij}}_{\text{nonsingular since inner product of basis}}$$

Proposition 6.1 (11.1)**.** Let (Φ', E') be another root system. Suppose it has base $\Delta' = \{\alpha'_1, \cdots, \alpha'_s\}$ such that $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$.
Then the bijection $\begin{matrix} \Delta & \to & \Delta' \\ \alpha_a & \mapsto & \alpha'_a \end{matrix}$ extends unique to a root system isomorphism $\phi : (\Phi, E) \to (\Phi', E')$.
Therefore, the Cartan matrix of Φ determines Φ upto isomorphism.

Proof. Δ, Δ' are both basis $\implies \exists$ vector space isomorphism $\phi : E \to E'$ such that $\alpha_a \to \alpha'_a$.
$\forall i, j \; \sigma_{\phi(\alpha_i)}(\phi(\alpha_j)) = \phi(\alpha_j) - \frac{\langle \phi(\alpha_j), \phi(\alpha_i) \rangle}{\langle \alpha_j, \alpha_i \rangle} \phi(\alpha_i) = \phi(\alpha_j - \langle \alpha_j, \alpha_i \rangle \alpha_i) = \phi(\sigma_{\alpha_i}(\alpha_j))$
Therefore, we have the commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\sigma_\alpha"] & E' \arrow[d, "\sigma_{\alpha'}"] \\ E \arrow[r, "\phi"] & E' \end{tikzcd} \qquad \text{(A)}$$

$\alpha \in \Delta$ and $\forall x \in E, \langle x, \alpha \rangle = \langle \phi(x), \phi(\alpha) \rangle \; (B)$
A and since $\mathcal{W}$ is generated by σ_α we have for all $\sigma \in \mathcal{W}$ we have:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\sigma"] & E' \arrow[d, "\sigma'"] \\ E \arrow[r, "\phi"] & E' \end{tikzcd}$$

Where $\sigma = \sigma_{\alpha_{l_1}} \cdots \sigma_{\alpha_{l_k}}$ and $\sigma' = \sigma_{\alpha_{l'_1}} \cdots \sigma_{\alpha_{l'_k}}$.
Therefore $\phi \circ \sigma \circ \phi^{-1} = \sigma' \in \mathcal{W} \quad (D)$.
Claim 1: $\phi(\Phi) \subseteq \Phi'$
Let $\beta \in \Phi \implies \exists \sigma \in \mathcal{W}, \exists \alpha \in \Delta$ such that $\sigma\alpha = \beta$
Also, $\phi(\beta) = \underbrace{(\phi \circ \sigma \circ \phi^{-1})}_{\in \mathcal{W}'} \underbrace{(\phi(\alpha))}_{\in \Phi'} \in \Phi'$
Claim 2: $\forall \beta, \gamma \in \Phi, \langle \gamma, \beta \rangle = \langle \phi(\gamma), \phi(\beta) \rangle$.
Since $\mathcal{W}$ is generated by reflections $\exists \sigma \in \mathcal{W}, \exists \alpha \in \Delta$ such that $\sigma\alpha = \beta$. Then,

$$\langle \gamma, \beta \rangle = \langle \sigma^{-1}\gamma, \alpha \rangle \underset{B}{=} \langle \phi \circ \sigma^{-1}(\gamma), \phi(\alpha) \rangle = \langle \sigma' \circ \phi \circ \sigma^{-1}(\gamma), \sigma' \circ \phi(\alpha) \rangle = \langle \phi(\gamma), \phi(\beta) \rangle$$

□

Definition. Fix $\Delta = \langle \alpha_1, \cdots, \alpha_l \rangle$ a base. The Coxeter graph of Φ consists of l vertices correlated to Δ and $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges between the i'th and j'th vertices $(i \neq j)$.

Remark. This graph is independent of Δ. Since Δ' another base $\implies \exists \sigma \in \mathcal{W}$ such that $\sigma\Delta = \Delta' \implies \langle \alpha_i, \alpha_j \rangle = \langle \sigma\alpha_i, \sigma\alpha_j \rangle$.
Also, we have $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle = 0, 1, 2, 3$.

e.g. [insert fig table]
We need more information to recover Cartan integer.
We cannot recover $\langle \alpha_i, \alpha_j \rangle$ when $\langle \rangle = 2, 3$.

Definition. The Dynkin diagram of $\Phi$is the Coxeter graph adding an arrow pointing to the shorter length root.
e.g [insert picture and matrix for B2 fig]

Proposition 6.2 (11.3)**.** Φ decomposes uniquely as the union of irreducible root system Φ_i in subspaces $E_i \subset E$ such that $E = E_1 \oplus \cdots \oplus E_t$ [orthogonal direct sum].

Proposition 6.3 (Ex 9.1)**.** $E' \subset E$: a subspace. If $\sigma_\alpha(E') \subseteq E'$ then either $\alpha \in E'$ or $E' \subseteq P_\alpha$

Proof. Suppose $E' \subsetneq P_\alpha$. Then $\exists \beta \in E'$ such that $(\alpha, \beta) \neq 0$.
$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in E' \implies \alpha \in E'$. ☐

Proof. (Of 11.3)
Δ cannot be partitioned into proper $\Delta_1 \cup \Delta_2$ such that $(\Delta_1, \Delta_2) = 0 \iff$ the coxeter graph of Φ is connected (A).
[insert picture fig]
Assume that the coxeter graph consists of connected components $C_1, \cdots, C_t$.
Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_t$ be the partition so that Δ_i correlates to C_i.
$A \implies \Delta_i$ are mutually orthogonal. Define $E_i = \text{span}_{\mathbb{R}}(\Delta_i)$.
Therefore, $E = E_1 \oplus \cdots \oplus E_t$ [orthogonal]. (B)
Define $\Phi_i = \{\alpha \in \Phi \mid \alpha : \text{a } \mathbb{Z}\text{-linear combination of } \Delta_i\}$.
Then, (Φ_i, E_i) is a root sytem [by checking the axioms].
$\Delta_i \subseteq \Phi_i$ is a base (again checking the axioms).
Therefore, Φ_i are irreduible.
Finally, we want to check $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$.
The Weyl group $\mathcal{W}_i$ corresponding to Φ_i is the subgroup of $\mathcal{W}$ generated by σ_α where $\alpha \in \Delta_i$.
B $\implies \sigma_\alpha$ acts trivially on E_i for any $\alpha \in \Delta_j, i \neq j$. (C)
E_i is $\mathcal{W}_i$ invariant, so $\sigma_\alpha(E_i) \subseteq E_i \forall \alpha \in \Delta_i \overset{C}{\implies} E_i$ is $\mathcal{W}$-invariant.
Thus, $\sigma(E_i) \subseteq E_i$ for any $\sigma \in \mathcal{W}$
Now we use Exercise 9.1.
$\supseteq$ is trivial by definition.
$\subseteq$ $\alpha \in \Phi \underset{D}{\implies} \sigma_\alpha(E_i) \subseteq E_i \overset{\text{Ex 9.1}}{\implies} \exists 1 \leq j \leq t$ such that $\alpha \in E_j$ but $\alpha \notin E_i \forall i \neq j$.
Thus, $\alpha \in \text{span}_{\mathbb{R}}(\Delta_i) \cap \Phi \implies \alpha$ is a $\mathbb{Z}$-linear combination of Δ in $\text{span}(\Delta_i)$.
Therefore, α is a $\mathbb{Z}$-linear combination of Δ_i.
This is the definition of Φ_i.
Therefore, $\alpha \in \Phi_i$.

☐

Theorem 6.4 (Classification Theorem)**.** Let Φ be an irreducible root system. Then, Φ can only have $A - G$ type Dynkin diagrams.

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Theorem 7.1 (11.4)**.** If Φ is an irreducible root system, its Dynkin diagram is one of the following:

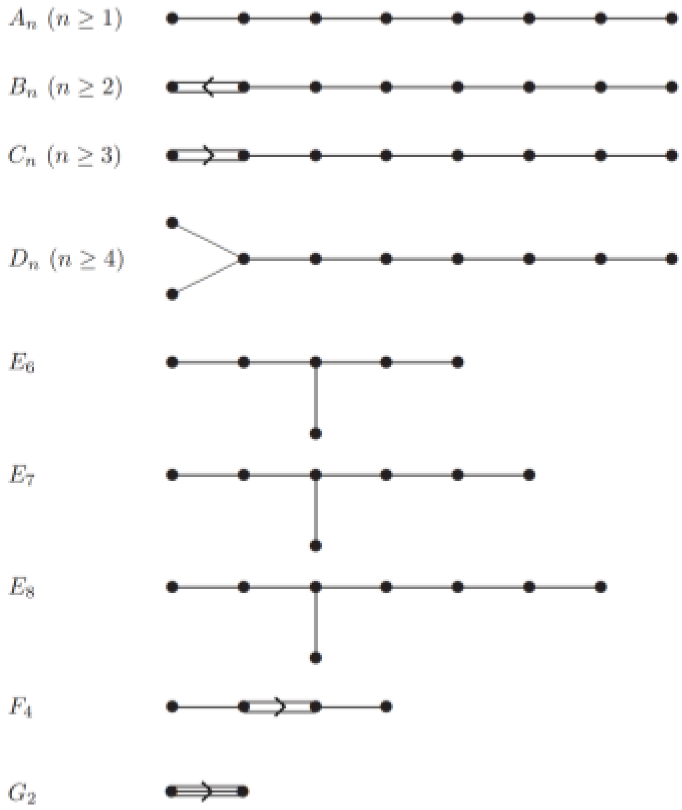

Figure 1: Dynkin Diagrams

Proof. Suffices to Show: Coxeter graph is one of the following:

$$A_l, B_l(= C_l), D_l, E_{6,7,8}, F_4, G_2$$

So we can ignore the length of the roots. We work with unit vectors.

Definition. Let E be euclidean space of dimension m and let $\mathfrak{A} = \{\varepsilon_1, \cdots, \varepsilon_n\} \subseteq E$ be admissible if:

1) Vectors are linearly independent.
2) Vectors are unit vectors.
3) $(\varepsilon_i, \varepsilon_j) \leq 0$ if $i \neq j$.
4) $4(\varepsilon_i, \varepsilon_j)^2 = 0, 1, 2$ or 3.

Then, The graph Γ of $\mathfrak{A}$[coxeter graph] has n vertices and $4(\varepsilon_i, \varepsilon_j)^2$ as edges.
Existence:

$$\Delta \supseteq \{\alpha_1, \cdots, \alpha_n\} \implies \left\{\frac{\alpha_1}{\sqrt{(\alpha_1, \alpha_1)}}, \cdots, \frac{\alpha_n}{\sqrt{(\alpha_n, \alpha_n)}}\right\} := \{\varepsilon_1, \cdots, \varepsilon_n\} \text{ admissible.}$$

$$4(\varepsilon_i, \varepsilon_j)^2 = \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_j)(\alpha_j, \alpha_i)} = \langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$$

1) A subset $\mathfrak{A}' \subseteq \mathfrak{A}$ is still admissible.

2) $\#\{$pairs of vertices in Γ connected by each other$\} < n$.

   Since set $\varepsilon := \sum_{i=1}^{n} \varepsilon_i \neq 0$. Thus $0 < (\varepsilon, \varepsilon) = \underbrace{n}_{=\sum_i (\varepsilon_i, \varepsilon_i)} + \sum_{i<j} 2(\varepsilon_i, \varepsilon_j)$.

   Suppose $(\varepsilon_i, \varepsilon_j) < 0$.

   $4(\varepsilon_i, \varepsilon_j)^2 = 1, 2, 3 \implies 2(\varepsilon_i, \varepsilon_j) \leq -1$.

3) Γ contains no cycles.

   Let $\Gamma' \subseteq \Gamma$ be a vertix on k vertices.

   Then $\#\{$such pairs from (2) in $\Gamma'\} \geq k$. Contradiction by 2.

4) $\#\{$edges originated at a vertex (ε) of $\Gamma\} \leq 3$

   Let $\varepsilon, \eta_1, \cdots, \eta_k \in \mathfrak{A}$ be distinct such that $(\varepsilon, \eta_i) \neq 0$.

   $(3) \implies (\eta_i, \eta_j) = 0 \forall i \neq j$.

   Set $\eta_0' := \epsilon - \sum_{i=1}^{k} (\varepsilon, \eta_i)\eta_i$ and $\eta_0 := \frac{\eta_0'}{(\eta_0', \eta_0')}$. Then,

   - $(\eta_0', \eta_i) = (\varepsilon, \eta_i) - (\varepsilon, \eta_i)(\eta_i, \eta_i) = 0 \implies (\eta_0, \eta_i) = 0$.
   - $\eta_0 \in \operatorname{span}\{\varepsilon, \eta_1, \cdots, \eta_k\} \implies (\eta_0, \varepsilon) \neq 0$.
   - $(\eta_0', \eta_0') = (\varepsilon, \eta_0') \implies \sum_{i=0}^{k} (\varepsilon, \eta_i)\eta_i = (\varepsilon, \eta_0)\eta_0 + \sum_{i=1}^{k} (\varepsilon, \eta_i)\eta_i$
     $= \frac{(\varepsilon, \eta_0')}{(\eta_0', \eta_0')}\eta_0' + \sum_{i=1}^{k} (\varepsilon, \eta_i)\varepsilon_i = \varepsilon$.
     $\implies 1 = (\varepsilon, \varepsilon) = \sum_{i=0}^{k} (\varepsilon, \eta_i)^2 > \sum_{i=1}^{k} (\varepsilon, \eta_i)^2 \implies 4 > \sum_{i=1}^{k} 4(\varepsilon, \eta_i)^2$.

5) The only graph Γ containing a triple edges is G_2.

6) Let $\{\varepsilon_1, \cdots, \varepsilon_k\} \subseteq \mathfrak{A}$ have a subgraph

   [insert pic like A1 with veps 1, ..., veps k]

   $\mathfrak{A}' = (\mathfrak{A} \setminus \{\varepsilon_1, \cdots, \varepsilon_k\}) \cup \{\varepsilon\}$ is admissible where $\varepsilon = \sum_{i=1}^{k} \varepsilon_i$.

   Condition 1 is satisfied automatically.

   Condition 2: ε is a unit: $2(\varepsilon_i, \varepsilon_{i+1}) = -1 \implies (\varepsilon, \varepsilon) = k + \sum_{i=1}^{k-1} (\varepsilon_i, \varepsilon_{i+1}) = 1$ so it holds.

   Condition 3: $\eta \in \mathfrak{U} \setminus \{\varepsilon_1, \cdots, \varepsilon_k\}$ can be connected to at most one of $\varepsilon_1, \cdots, \varepsilon_k$.
   $3 \implies (\eta, \varepsilon) = 0$ or $(\eta, \varepsilon) = (\eta, \varepsilon_j) \leq 0 \implies (\eta, \varepsilon) \leq 0$.

   Condition 4: $4(\eta, \varepsilon)^2 = 0$ or $4(\eta, \varepsilon_i)^2 = 0, 1, 2$ or 3

7) Γ contains no subgraph of the form:

   [insert figure]

8) Any connected Γ of an admissible set has one of the following forms:

   [insert figure]

9) The second type in 8: $B_l (= C_l), F_4$.

   Set $\varepsilon := \sum_{i=1}^{p} i\varepsilon_i, \eta := \sum_{i=1}^{q} i\eta_i$.

   Then $2(\varepsilon_i, \varepsilon_{i+1}) = -1 = 2(\eta_j, \eta_{j+1})$.

   Then, $(\varepsilon, \varepsilon) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = \frac{p(p+1)}{2}$, and $(\eta, \eta) = \frac{q(q+1)}{2}$. (A)

   $4(\varepsilon_p, \eta_q)^2 = 2 \implies (\varepsilon, \eta)^2 = (p\varepsilon_p, q\eta_q)^2 = \frac{p^2 q^2}{2}$ (B).

   The Schwarz inequality of linear independent $\varepsilon \cdot \eta : (\varepsilon, \eta)^2 < (\varepsilon, \varepsilon)(\eta, \eta)$.

   A and B $\implies \frac{2p^2q^2}{2} < \frac{pq(p+1)(q+1)}{4} \implies pq - p - q - 1 < 0 \implies (p-1)(q-1) < 2$
   $\implies \underbrace{p = q = 2}_{F_4}$ or $\underbrace{p = 1 \text{ or } q = 1}_{B_l}$.

10) The 3rd type in 8: D_l, E_6, E_7, E_8.
Set up $\eta^i := \sum_j \eta^i_j \implies (\eta^i, \eta^j) = 0, (\varepsilon, \eta^i) \neq 0$.
Applying the method in the proof of (4),
$\eta^{0\prime} := \varepsilon - \sum_{i=1}^{3} \frac{(\varepsilon,\eta^i)}{(\eta^i,\eta^i)}\eta^i$ and $\eta^0 := \frac{\eta^{0\prime}}{\sqrt{(\eta_0', \eta_0')}}$
It is easy to show that:

- $(\eta^0, \eta^i) = 0$
- $(\varepsilon, \eta^0) \neq 0$
- $\varepsilon = (\varepsilon, \eta^0)\eta^0 + \sum_{i=1}^{3} \frac{(\varepsilon,\eta^i)}{(\eta^i,\eta^i)}\eta^i$

Then $1 = (\varepsilon, \varepsilon) > \sum_{i=1}^{3} \frac{(\varepsilon,\eta^i)^2}{(\eta^i,\eta^i)}$ by the same reason in (4).
$(\eta', \eta') = \frac{p(p-1)}{2} \implies \frac{(\varepsilon,\eta')^2}{(\eta',\eta')} = \frac{p-1}{2p} = \frac{1}{2}\left(1 - \frac{1}{p}\right)$. Apply η^2, η^3.
$\implies 1 > \sum \frac{(\varepsilon,\eta_i)^2}{(\eta^i,\eta^i)} = \frac{1}{2}\left(3 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right) \implies \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ (C).
WLOG assume $\frac{1}{p} \leq \frac{1}{q} \leq \frac{1}{r} \leq \frac{1}{2}$ (D) [if one of them is 1 graph is A_l].
C, D $\implies \frac{1}{3} < \frac{1}{r} \leq \frac{1}{2} \implies r = 2 \implies \frac{1}{p} + \frac{1}{q} > \frac{1}{2}$ (E).
D, E $\implies \frac{1}{4} < \frac{1}{q} \leq \frac{1}{2} \implies q = 2, 2 \leq p \implies D_l$ or $q = 3, p = 3, 4, 5 \implies E_6, E_7, E_8$.

<u>Construction</u>:
[matrices for A_l, B_l, C_l, D_l]
Let $E = \mathbb{R}^n$ with the usual inner product and standard basis $\{\varepsilon_1, \cdots, \varepsilon_n\}$. Let $I = \{\mathbb{Z}$-linear combination of $\{\varepsilon_1, \cdots, \varepsilon_n\}\}$.
$A_l (l \geq 1) E$: a l-subspace of $\mathbb{R}^{l+1}$ orthogonal to $\varepsilon_1 + \cdots + \varepsilon_{l+1}$.
$I' = I \cap E$.
$\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 2\} = \{(\varepsilon_i - \varepsilon_j), i \neq j\}$.
Root system: $\Delta = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_l - \varepsilon_{l+1}\}$
Weyl Group: $\sigma_{\varepsilon_i - \varepsilon_{i+1}} : \begin{matrix} \varepsilon_i \mapsto \varepsilon_{i+1} \\ \varepsilon_{i+1} \mapsto \varepsilon_i \end{matrix} \leftrightarrow (i, i+1)$.
Thus, $\mathcal{W} \cong S_{l+1}$.
$B_l (l \geq 2) E = \mathbb{R}^l$
$\Phi_B = \{\alpha \in I \mid (\alpha, \alpha) = 1 \text{ or } 2\} = \{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j), i \neq j\}$.
$\Delta = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l\}$ a base.
Weyl Group $\mathcal{W}$:
$\sigma_{\varepsilon_i - \varepsilon_{i+1}} \leftrightarrow (i, i+1)$
$\sigma_{\varepsilon_l} \varepsilon_l \mapsto -\varepsilon_l$
$\mathcal{W} \cong (\mathbb{Z}/2\mathbb{Z})^l \rtimes S_l$.
$D_l (l \geq 4) E = \mathbb{R}^l, \Phi = \{\alpha \in I \mid (\alpha, \alpha) = 2\} = \{\pm(\varepsilon_i \pm \varepsilon_j)\}$.
$\Delta = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_{l-1} + \varepsilon_l\}$
$\sigma_{\varepsilon_i - \varepsilon_{i+1}} : (i, i+1)$
$\sigma_{\varepsilon_{l-1} + \varepsilon_l}$: $\varepsilon_{l-1} \mapsto -\varepsilon_l, \varepsilon_l \mapsto -\varepsilon_{l-1}$.
$\mathcal{W} \cong (\mathbb{Z}/2\mathbb{Z})^{l-1} \rtimes S_l$.

□

7.1 12.2 Automorphism

Claim 1: $\operatorname{Aut}\Phi \cong \Gamma \rtimes \mathcal{W}$ where $\Gamma = \{\tau \in \operatorname{Aut}\Phi \mid \tau(\Delta)\Delta\}$.
Claim 2: Γ may be identified with automorphims of its dynkin diagram of Φ.

8 Thursday, 12/5/2024, Cartan Subalgebras (CSA) by Rostyslav

2+2 Definitions.
General Lie Algebra:

i) In terms of a normalizer

ii) In terms of Engel subalgebra

Semisimple Lie Algebra:

i) Maximal toral subalgebras

ii) Maximal centralizer

If $x \in \operatorname{End}(V), V = \bigoplus V_a(x), V_a(x) = \ker(x - a\,\mathrm{id})^m$ where m is a multiplicity of root of char. poly.
We have $x\big|_{v_a(x)} = \underbrace{a}_{\text{scalar}} + \underbrace{n}_{\text{nilpotent}}$.
Then, $L = \sqcup_{a \in F} L_a(\operatorname{ad} x) = L_u(\operatorname{ad} x) \oplus L_n(\operatorname{ad} x)$

Lemma 8.1. If $a, b \in F$ then,

$$[L_a(\operatorname{ad} x), L_b(\operatorname{ad} x)] \subset L_{a+b}(\operatorname{ad} x)$$

In particular, when $L_t(\operatorname{ad} x)$ is a subalgebra of $L, \operatorname{char} F = 0, a \neq 0$ then,

$$\forall l \in L_a(\operatorname{ad} x)\, l \text{ is nilpotent}$$

Fact: (simple) adjoints are derivations

$$(\operatorname{ad} x - (a+b))^m [yz] = \sum_{i=0}^{m} \binom{m}{i} [(\operatorname{ad} x - a)^i (y), (\operatorname{ad} x - b)^{m-i}(z)]$$

For m sufficiently large all elements on the right well vanish.

Definition (Engel Subalgebra)**.** is $L_0(\operatorname{ad} x)$.

Lemma 8.2 (15.2.A)**.** Let $K \subset L$-subalgebra. Choose $z \in K$ such that $L_0(\operatorname{ad} z)$ is minimal among $L_0(\operatorname{ad} x)$ for all $x \in K$. Suppose $K \subset L_0(\operatorname{ad} z)$. Then $L_0(\operatorname{ad} z) \subset L_0(\operatorname{ad} x)\, \forall x \in K$.

Let $x \in K$ fixed, but arbitrary. Consider a family of endomorphisms of $L\{\operatorname{ad}(z + cx) \mid c \in F\}$. $K_0 = L_0(\operatorname{ad} z)$ is a subalgebra of L including K therefore these endoomorphisms will stabilize $K_0 (= L_0(\operatorname{ad} z))$.
Can induce endomorphism of $L\big|_{K_0(=L_0(\operatorname{ad} z))}$.
Endomorphism is $\operatorname{ad}(z + cx)$. Let f, g be char. poly in K_0 over L/K and let $n = \dim K_0, n - \dim L$.

$$f(T, c) = T^r + f_1(c) T^{n-1} + \cdots + f_n(c)$$

$$g(T, c) = T^{n-r} + g_1(c) + T^{n-r-1} + \cdots g_{n-r}(c)$$

f_i, g_i are polynomials.
By definition, eigenvalue 0 appears only in K_0 when $T = 0, g_{n-r}$ is not identically zero on F. Lets take $c_1, \cdots, c_{r+1}$ not zeros of g_{n-r}. To say $g_{n-r}(0) = 0 \iff 0$ is not an eigenvalue of $\operatorname{ad}(z + cx)$ on the quotient space.
$\implies \forall L_0(\operatorname{ad}(z + cx))$ lie in the subspace of K_0.
But K_0 is minimal.
$K_0 = L_0(\operatorname{ad} z) = L_0(z + c_i x)\, 1 \leq i \leq r + 1$.
The only eigenvalues $\operatorname{ad}(z + cx)$ has is 0.

$\implies f(T, c_i) = T^r, f_i = f(T, c_i)$.
$\forall f_i$ has $r + n$ distinct zeros $\implies \forall f_i$ are identically zero.
$L_0(\operatorname{ad}(z + cx)) \supset K_0 \forall c \in F$.
Replace x with $x - z, c = 1$.
$L_0(\operatorname{ad} x) \supset K_0 = L_0(\operatorname{ad} x)$ so we're done.

Lemma 8.3 (15.2.B)**.** If $K \subset L$ subalgebra and $L_0(\operatorname{ad} x) \subset K \implies K$ is self normalizing. In particular, Engel subalgebras are self-normalizing.

$L_0(\operatorname{ad} x) \subset K \implies \operatorname{ad} x$ acts on $N_L(K)/K$ without an eigenvalue 0 in $\ker = 0$.
$x \in K\ [N_L(K)x] \subset K \implies \operatorname{ad} x$ acts trivially on $N_L(K)/K \implies N_L(K) = K$.

Definition (Cartan Subalgebra)**.** Cartan Subalgebra is a nilpotent subalgebra of L that is self-normalizing.

Theorem 8.4 (15.3)**.** Let H-subalgebra of L. H is a CSA $\iff$ H is a minimal Engel subalgebra.

$\impliedby$: Assume it is a minimal Engel subalgebra. Then, $H = L_0(\operatorname{ad} z) \overset{15.2.B}{\implies} H$ is self-normalizing. $\overset{15.2.A}{\implies} L_0(\operatorname{ad} z) \subset L_0(\operatorname{ad} x)$.
We apply Engel's Theorem which states if $\forall x \in L \operatorname{ad} x$ is nilpotent then L is nilpotent.
$\forall x \in H$ in particular $\operatorname{ad}_H x$ is nilpotent so H is nilpotent.
$\implies$: Let H be CSA, H is nilpotent by defintion. $H \subset L_0(\operatorname{ad} x)$. We want to prove $\exists z$ such that $H = L_0(\operatorname{ad} z)$.
Lets assume that is not the case.
Take $L_0(\operatorname{ad} z)$ smallest, $\overset{15.2.A}{\implies} L_0(\operatorname{ad} z) \subset L_0(\operatorname{ad} x)$.
$L_0(\operatorname{ad} z)/H$ here $x \in H$ will act as a nilpotent.
H annihilates some $y + H$ where $y \neq 0$.
$\exists y \in H$ such that $[Hy] \subset H$. But H is self normalizing. Contradiction! So we're done.