Partial Differential Equation 2 MATH 541

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This is dedicated to Sobolev Spaces, which we apply to Elliptic (linear) PDE.
Last part of the course is in a different direction. We talk about applying this to Parabolic/Hyperbolic PDE.
Also Schauder Theory

Monday, 8/26/2024

There are very explicit formula for certain PDE. For example D'Alembert, Poisson etc.
Weak Solutions to PDE: There is some notion of solution that doesn't have the requisite number of derivatives in the classical calculus sense. So we lower our notion of what a solution is.
ex. Conservation law: Burger's Equation: $u_t + uu_x = 0, u(x, 0) = u_0(x)$. If we try to solve this for $-\infty < x < \infty, t > 0$, we have method of characteristics that attempts to give us explicit formula (solution will be constant along lines of slope $\frac{1}{u_0}$), but there is trouble for general u_0. These lines an bump into each other, so in that point our solution has to equal two different numbers.

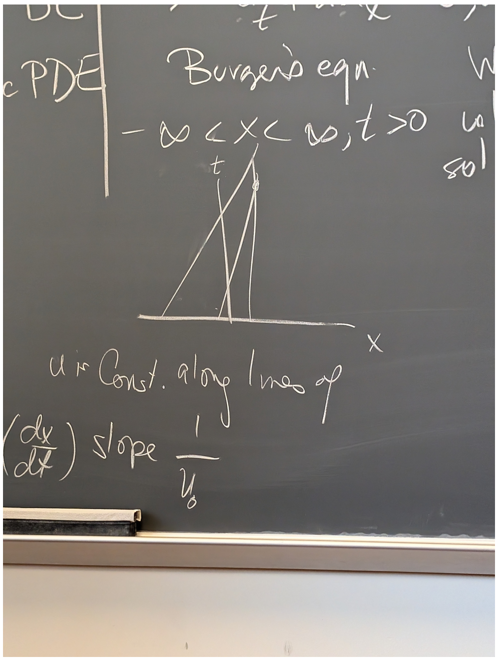

Figure 1: Burger's Equation

Can also happen that a classical solution does exists, BUT it is easiest to find a weak solution.
- Find a weak solution
- Show it's classical (regularity theory)
For linear elliptic PDE's ex. Laplace's Equation,

$$\Delta u = u_{x_1x_1} + \cdots + u_{x_nx_n} = 0$$

in $\Omega \subset \mathbb{R}^n$
A weak solution satisfies the equation obtained through multiplication by a test function C_c^∞ and integrate by parts
ie take any $\phi \in C_c^\infty(\Omega)$
$\int_\Omega \phi \Delta u = 0$
$= -\int_\Omega \nabla\phi \cdot \nabla u + \int_{\partial\Omega} \phi \nabla u \cdot \nu$
Note that the second term is integrated over the boundary. So it goes to 0
So, if:

$$\int_\Omega \nabla\phi \cdot \nabla u = 0 \forall \phi \in C_c^\infty$$

We say u weakly solves Laplace's equation because it requires only one derivative (not two).

Definition 1 (Weak Derivative)**.** A locally integrable function u [notationally $u \in L^1_{loc}(\Omega)$] has a weak x_i derivative v if v is locally integrable and
$\int_\Omega u\phi_{x_i}\,\mathrm{d}x = -\int_\Omega v\phi\,\mathrm{d}x$ for all $\phi \in C_c^\infty(\Omega)$

u and v can be terrible near the boundary but ϕ vanishes so we don't care!
Recall: Multi-inde notation
If $u : \mathbb{R}^n \to \mathbb{R}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ where α_j is non-negative integer then,

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}$$

eg in $\mathbb{R}^3$ for $\alpha = (1, 2, 1)$ then $D^\alpha u = u_{x_1 x_2 x_2 x_3}$

Definition 2. Given $u, v \in L^1_{loc}(\Omega), \Omega \subset \mathbb{R}^n$ and α a multi-index, we say v is the weak αth derivative of u if integration by parts works:
if $\int_\Omega u D^\alpha \phi\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\phi\,\mathrm{d}x, \forall \phi \in C_c^\infty(\Omega)$

If we are define weak anything, if our weak thing actually happened to be good, we want it to satisfy the strong definition as well! For example if differentiation is allowed, then integration by parts would actually work. So, if u were smooth, then our derivative would actually satisfy the solution, since

$$\int_\Omega (D^\alpha u - v)\phi\,\mathrm{d}x = 0$$

So $D^\alpha u = v$ almost everywhere.
Recall: $u \in L^p(\Omega), p > 0$ if $\int |u|^p\,\mathrm{d}x < \infty$
$u \in L^p_{loc}(\Omega)$ if $\forall \Omega_1 \subset\subset \Omega, \int_{\Omega_1} |u|^p\,\mathrm{d}x < \infty$

Sobolev Spaces

Definition 3 (Sobolev Spaces)**.** Fix $1 \le p \le \infty$ and a non-negative integer k. Let $\Omega \subset \mathbb{R}^n$ be open.
Then the Sobolev space $W^{k,p}(\Omega)$ consists of all functions $u \in L^1_{loc}(\Omega)$ such that for every multi-index α with $|\alpha| < k$, $D^\alpha u$ exists weakly and lies in $L^p(\Omega)$.

Example 1: Consider $\tilde{u}(x) = |x|$. Doesn't have a derivative. What about weak derivative?
Claim: u has weak 1st derivative $v(x) = \begin{cases} -1, & \text{if } x < 0; \\ 1, & \text{if } x > 0; \end{cases}$
We verify that using test function.
Let $\phi \in C_c^\infty(\mathbb{R})$

$$LHS = \int_{-\infty}^{\infty} v(x)\phi(x)\,\mathrm{d}x = -\int_{-\infty}^{0} \phi(x)\,\mathrm{d}x + \int_0^\infty \phi(x)\,\mathrm{d}x$$

$$RHS = -\int_{-\infty}^{\infty} \tilde{u}(x)\phi'(x)\,\mathrm{d}x = \int_{-\infty}^{0} x\phi'(x)\,\mathrm{d}x - \int_0^\infty x\phi'(x)\,\mathrm{d}x$$

By applying IBP

$$RHS = -\int_{-\infty}^{0} \phi(x)\,\mathrm{d}x + \int_{0}^{\infty} \phi(x)\,\mathrm{d}x$$

[boundary terms don't matter because either x or ϕ vanishes]
Since $|x|$ is locally integrable for any p, $\tilde{u} \in W^{1,p}_{loc}(\mathbb{R})$ for all p
Example 2: Consider the Heaviside function

$$u(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x < 0; \end{cases}$$

Is NOT going to be weakly differentiable!

Wednesday, 8/28/2024

Sobolev Space $W^{k,p}(\Omega)$

$u \in W^{k,p}(\Omega)$ if
$D^\alpha u \in L^p$ weakly for all α such that $|\alpha| \leq k$
This is a normed space.

$$\|u\|_{W^{k,p}(\Omega)} = \left[\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p\right]^{\frac{1}{p}}$$

If $p = \infty$ we just take the sup norm.
We have convergence: $\{u_j\} \subset W^{k,p}(\Omega)$ converges in $W^{k,p}$
to $u \in W^{k,p}(\Omega)$ if $|u_j - u|_{W^{k,p}(\Omega)} \to 0$ as $j \to \infty$

Definition 4. $W^{k,p}_0(\Omega) :=$ closure in $W^{k,p}(\Omega)$-norm of $C^\infty_o(\Omega)$

<u>Remark</u>: $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$ are Banach spaces [normed, complete [since Lp is complete]]
For $p = 2$ we don't usually use $W^{k,2}(\Omega)$. We use $H^k(\Omega)$. H is for Hilbert. This is a Hilbert space [there exists an inner product].

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v$$

Hölder's inequality implies:

$$\left|\int_\Omega D^\alpha u D^\alpha v\right| \leq \|D^\alpha u\|_{L^2} \|D^\alpha v\|_{L^2}$$

Let's go back to the example.
Let $u(x)$ be the heaviside function, 0 for negative, 1 for positive. Is it weakly differentiable? Is it in some sobolev space?
Is $u \in W^{k,p}(\mathbb{R})$ for some k and p? Does u have a weak solution?
Answer: <u>No</u>! We use contradiction.
Suppose there is such a $v \in L^1_{loc}(\mathbb{R})$ such that IBP holds:

$$\int_\mathbb{R} u\phi'\,\mathrm{d}x = -\int_\mathbb{R} v\phi\,\mathrm{d}x$$

for all $\phi \in C^\infty_c(\mathbb{R})$
Then, $\int_0^\infty \phi'(x)\,\mathrm{d}x = -\int_{-\infty}^\infty v\phi\,\mathrm{d}x$
Therefore, $\phi(0) = \int_{-\infty}^\infty v(x)\phi(x)\,\mathrm{d}x$
We use this for contradiction. Consider a sequence $\{\phi_j\}$ of test functions so that $\phi_j(0) = 1$ for all j and $0 \leq \phi_j(x) \leq 1$ for all x.
Further suppose that the support for ϕ_j shrinks to the origin.

$$\phi_j(0) = 1 = -\int_{\mathbb{R}} v\phi_j \,\mathrm{d}x$$

Now, $v\phi_j \to 0$ pointwise a.e.
$|v\phi_j| \leq |v| \in L^1_{loc}$
This gives us the desired contradiction.
Notice: $\phi(0) = \int v(x)\phi(x)$ is true for the 'function' dirac delta. This is not really a function, this is a distribution.
Moral of the story: We can't have too big of a discontinuity [here we have a jump discontinuity] and still be in Sobolev spaces.
Example: $f(x) = \frac{1}{|x|^\alpha}$ for $x \in B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$ and $\alpha > 0$
For which k, p, n, α is $f \in W^{k,p}(B(0,1))$?
Question 1: Is this in any Lp space? If no then game over.
Is $f \in L^p(B(0,1))$

$$\int_{B(0,1)} \frac{1}{|x|^{\alpha p}} \,\mathrm{d}x = \int_0^1 \int_{\partial B(0,r)} \frac{1}{|x|^{\alpha p}} \,\mathrm{d}S \,\mathrm{d}r$$

$$= \int_0^1 \frac{1}{r^{\alpha p}} \mu(\partial B(0,r)) \,\mathrm{d}r$$

$$= \omega_n \int_0^1 \frac{1}{r^{\alpha p}} r^{n-1} \,\mathrm{d}r = \omega_n \int_0^1 r^{-\alpha p + n - 1} \,\mathrm{d}r$$

So, $f \in L^p$ if $\alpha p \leq n$
Note that $|D^\alpha f| \leq L^p$ for $|\alpha| = 1$ provided $(\alpha + 1)p \leq n$
Is f weakly differentiable for $|\alpha| = 1$?
Consider $\epsilon < 1$

$$\int_{B(0,1)\setminus B(0,\epsilon)} f\phi_{x_i} \,\mathrm{d}x = \int_{B(0,1)\setminus B(0,\epsilon)} \nabla \cdot (0, 0, f\phi, 0, 0) \,\mathrm{d}x - \int_{B(0,1)\setminus B(0,\epsilon)} \phi f_{x_i} \,\mathrm{d}x$$

$$\int_{B(0,1)\setminus B(0,\epsilon)} f\phi_{x_i} \,\mathrm{d}x = -\int_{B(0,1)\setminus B(0,\epsilon)} \phi f_{x_i} \,\mathrm{d}x + \int_{\partial(B(0,1)\setminus B(0,\epsilon))} f\phi\nu_i \,\mathrm{d}S$$

Note that the outer normal dissapears, we only have the inner normal.

$$\int_{\partial(B(0,1)\setminus B(0,\epsilon))} f\phi\nu_i \,\mathrm{d}S = \int_{\partial B(0,\epsilon)} f\phi\nu_i \,\mathrm{d}S$$

Setting $\epsilon \to 0$ we see that the integral converges. Also,

$$\left| \int_{\partial(B(0,1)\setminus B(0,\epsilon))} f\phi\nu_i \,\mathrm{d}S \right| \leq \int_{\partial B(0,\epsilon)} |f\phi\nu_i| \leq c \int_{\partial B(0,\epsilon)} |f| \,\mathrm{d}S \leq \frac{c}{\epsilon^\alpha} \omega_n \epsilon^{n-1} \to 0$$

provided $n - 1 \geq \alpha$
So,

$$\int_{B(0,1)} f\phi_{x_i} \,\mathrm{d}x = -\int_{B(0,1)} \phi f_{x_i} \,\mathrm{d}x$$

So f is weakly differentiable for $|\alpha| = 1$
(See Appendix 5 in Evans)
Mollification:
There are lot of situation in PDE where you have a function and you don't know how nice is it in terms of derivative. So you convolve it so that it is nice and take some limit.
Let η satisfy $\eta \in C_c^\infty(\mathbb{R}^n)$
Suppose $\eta \equiv 0$ for $|x| \geq 1$
$\int_{\mathbb{R}^n} \eta(x) \,\mathrm{d}x = 1$

Suppose η is radial, $\eta = \eta(|x|)$ just a function of radial distance
Note that there is no such analytic function. But there are infinitely differentiable ones.
Define $\eta_\epsilon \coloneqq \epsilon^{-n}\epsilon(\frac{x}{\epsilon})$
So we rescale the function.
Then given $u \in L^1_{loc}(?)$ we define the mollification

$$u_\epsilon(x) = u * \eta_\epsilon = \int u(x-y)\eta_\epsilon(y)\,\mathrm{d}y = \int u(y)\eta_\epsilon(x-y)\,\mathrm{d}y$$

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Given $\Omega \subset \mathbb{R}^n$, open, bounded
$\forall \epsilon$ define $\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \epsilon\} = \Omega_\epsilon$
Let $\eta \in C_c^\infty(\mathbb{R}^n)$ such that
$0 \le \epsilon \le 1$ and $\int_{B(0,1)} \eta(x)\,\mathrm{d}x = 1$ and $\mathrm{supp}(\eta) \subset \overline{B(0,1)}$
Define $\eta_\epsilon(x) \coloneqq \epsilon^{-n}\eta(\frac{x}{\epsilon})$
Then, $\mathrm{supp}(\eta_\epsilon) \subset \overline{B(0,\epsilon)}$

$$\int_{B(0,\epsilon)} \eta_\epsilon(x)\,\mathrm{d}x = \epsilon^{-n}\int_{B(0,\epsilon)} \eta(\frac{x}{\epsilon})\,\mathrm{d}x = \int_{B(0,1)} \eta(x)\,\mathrm{d}x = 1$$

Theorem 1. Given $f \in L^1_{loc}(\Omega)$ define $f_\epsilon \coloneqq \eta_\epsilon * f$ for $x \in \Omega_\epsilon$

$$f_\epsilon(x) = \int_\Omega \eta_\epsilon(x-y)f(y)\,\mathrm{d}y$$

Then, i: $f_\epsilon(x)$ is infinitely differentiable.
ii: $f_\epsilon \to f$ pointwise a.e.
iii: If f is continuous then $f_\epsilon \to f$ uniformly on compact subsets of Ω
iv: If for $1 \le p < \infty, f \in L^p(\Omega)$ then $f_\epsilon \to f$ in $L^p(\Omega)$. Also true for $f \in L^p_{loc}$

Proof. (i): Convolution is a child, and it inherits the nicest properties of the parent. η_ϵ is nicer.
Idea: it is legal to bring derivatives inside the integral.

$$\frac{f_\epsilon(x+he_i) - f_\epsilon(x)}{h} = \int_\Omega \left[\frac{\eta_\epsilon(x+he_i-y) - \eta_\epsilon(x-y)}{h}\right] f(y)\,\mathrm{d}y$$

Stuff in brackets converges uniformly in y to $\eta_{\epsilon_{x_i}}(x-y)$ using Taylor remainder theorem.
Then Lebesgue Dominated Convergence finishes the job.
(iii) Fix $K \subset \Omega$ where K compact.
$\forall x \in K$ we have $|f_\epsilon(x) - f(x)| = \left|\int_\Omega(\eta_\epsilon(y)f(x-y))\,\mathrm{d}y - f(x)\right|$

$$= \left|\int_\Omega \eta_\epsilon(y)\left[f(x-y) - f(x)\right]\,\mathrm{d}y\right|$$

$$\le \int_\Omega \eta_\epsilon(y)\left|f(x-y) - f(x)\right|\,\mathrm{d}y$$

Since K is compact, f is uniformly continuous on K.
So, $\forall \beta > 0$ we have ϵ_0 such that $\forall \epsilon < \epsilon_0$ such that $\forall x, y$ such that $|y| < \epsilon$ we have $|f(x-y) - f(x)| < \beta$
So,

$$\le \int_\Omega \eta_\epsilon(y)\beta\,\mathrm{d}x = \beta$$

□

What is the relevance to Sobolev spaces?
Often we are going to try to prove some estimates. We want to prove some inequalities. Then, we mollify and prove it for the mollification. If we mollify a Sobolev function, we get an approximation.

Theorem 2 (Local Approximation Away from $\partial\Omega$)**.** Assume $u \in W^{k,p}(\Omega)$ [aka function has k'th weak derivatives which are locally integrable in order p]. Define $u_\epsilon = \eta_\epsilon * u$ in $\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$
THen, $u_\epsilon \in C^\infty(\Omega_\epsilon)$
And $u_\epsilon \to u$ in $W^{k,p}_{loc}(\Omega)$

Proof. Note infinite derivative we already have by property of mollification.
Fix the multi-index α suh that $|\alpha| \le k$.
<u>Claim 1</u>: $D^\alpha u_\epsilon = \eta_\epsilon * D^\alpha u$
In other words, mollification and derivatives commute.
To see this, consider $x \in \Omega_\epsilon$

$$D^\alpha u_\epsilon = D^\alpha \int_\Omega \eta_\epsilon(x-y)u(y)\,\mathrm{d}y$$

$$= \int_\Omega D^\alpha_x \eta_\epsilon(x-y)u(y)\,\mathrm{d}y$$

$$= (-1)^{|\alpha|} \int_\Omega D^\alpha_y \eta_\epsilon(x-y)u(y)\,\mathrm{d}y$$

$$= (-1)^{|\alpha|}(-1)^{|\alpha|} \int_\Omega \eta_\epsilon(x-y)D^\alpha u(y)\,\mathrm{d}y$$

$$= \eta_\epsilon * D^\alpha u$$

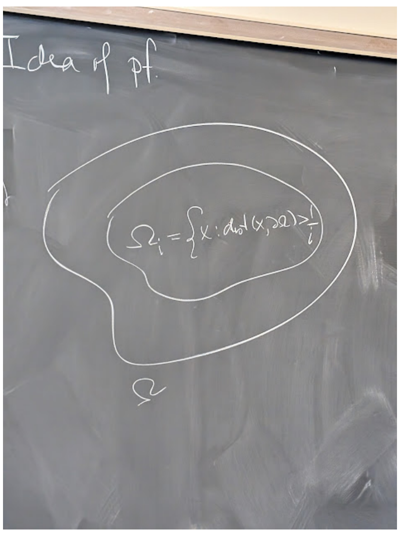
proving the claim.
Now, fix $V \subset \Omega$ with open $\overline{V} \subset \Omega$ ($V \subset\subset \Omega$)
Apply previous theorem, item iv and the claim
$D^\alpha u_\epsilon \to D^\alpha u$ in $L^p(V)$ for all α such that $|\alpha| \le k$

□

Theorem 3 (Global Approximation)**.** Theme:
Sobolev can be approximated by smooth sobolev
Let $\Omega \subset \mathbb{R}^n$ be open, bounded. Assume $u \in W^{k,p}(\Omega)$ for $1 \le p < \infty$
Then, $\exists\{u_m\} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ [infinitely smooth AND sobolev] such that $u_m \to u$ in $W^{k,p}(\Omega)$. Meaning:

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(\Omega)} \to 0$$

Idea of Proof:



We have a bunch of Ω_i with $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$
Use a partition of unity $\{\phi_i\}$ so that $\sum \phi_i(x) = 1$, mollify.
Proof is on Evans.

Wednesday, 9/4/2024

Traces

We are going to solve PDEs using Sobolev spaces. We typically specify boundary conditions in PDEs. But in first glance, since Sobolev functions are defined upto a set of measure zero, it seems ill suited to dealing with boundary conditions [since boundary $\partial\Omega$ has measure 0].
How to define boundary values for a Sobolev functions?
We are going to establish an inequality for smooth function.

Theorem 4. Assume $\partial\Omega$ is C^1 surface[1]. Then $\exists$ a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ for $1 \le p < \infty$ such that:

i) $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$

ii) $\|Tu\|_{L^p(\partial\Omega)} \le C \|u\|_{W^{1,p}(\Omega)}$ for some $C = C(p, \Omega)$

Proof. Outline:

1) Fix $x_0 \in \partial\Omega$. Suppose $\partial\Omega$ is 'flat' near x_0. In fact, define ball B centered at x_0 such that inside B, $\partial\Omega$ is flat.

   [insert picture]

   Then we can define $x' = (x_1, \cdots, x_{n-1})$. If the last coordinate is positive, we are inside Ω.

   Inside B, $\partial\Omega$ is flat. we can define $B' \subset B$ centered at x_0.

   Let $\zeta \in C_0^\infty(B)$ so that $\zeta \equiv 1$ on B' and $\zeta \ge 0$.

   Define $\Gamma \coloneqq \partial\Omega \cap B'$.

   Assume $u \in C^1(\overline{\Omega})$.

$$\int_\Gamma |u|^p \,\mathrm{d}x' \le \int_{\{x_n=0\}\cap B} \zeta |u|^p \,\mathrm{d}x'$$

$$= -\int_{\{x_n=0\}\cap B} (0, \cdots, 0, \zeta|u|^p) \cdot (0, \cdots, 0, -1) \,\mathrm{d}x'$$

   We are almost set up for divergence theorem. Since $\zeta = 0$ on the boundary of B, we actually have the whole boundary! Applying the divergence theorem, we get:

$$= -\int_{B\cap\Omega} \frac{\partial}{\partial x_n} [\zeta(x)|u(x)|^p] \,\mathrm{d}x$$

$$= -\int_{B\cap\Omega} \left[\zeta_{x_n}|u|^p + \zeta p|u|^{p-1}\frac{u}{|u|}u_{x_n}\right] \mathrm{d}x$$

$$= -\int_{B\cap\Omega} \left[\zeta_{x_n}|u|^p + \zeta p|u|^{p-1}\operatorname{sgn}(u) u_{x_n}\right] \mathrm{d}x$$

   Now we use <u>Young's Inequality</u>.

[1] meaning $\partial\Omega$ can be described as the graph of a C^1 function.

Theorem 5 (Young's Inequality)**.** For any $a, b > 0$ and p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, we have:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

We are estimating abolute value, so we can take absolute value. Using Young's inequality on $a = |u_{x_n}|$ and $b = |u|^{p-1}$,

$$\int_{\Gamma} |u|^p \, \mathrm{d}x'$$

$$\leq \int_{B}^{\infty} \left[|\zeta_{x_n}||u|^p + p|\zeta| \left[\frac{(|u|^{p-1})^{p/(p-1)}}{p/(p-1)} + \frac{|u_{x_n}|^p}{p} \right] \right] \mathrm{d}x$$

$$\leq C \int_{B \cap \Omega} [|u|^p + |\nabla u|^p] \, \mathrm{d}x \leq C \int_{\Omega} [|u|^p + |\nabla u|^p] \, \mathrm{d}x$$

Now, $\partial\Omega \in C^1$ means, centered at any point $x_0 \in \partial\Omega$ $\partial\Omega$ can be written as a graph $x_n = f(x')$ where $|f|_{C_1}$ is bounded.[2]

Therefore, Ball inside of which $\partial\Omega$ is a graph has radius R depending on C^1 norm.

2) For the next step, we flatten the boundary of Ω by a change of variables.

[insert picture]

[insert picture 2]

Change variables: $y = (y', y_n)$.

Set $y' = x'$

$y_n := x_n - f(x')$

$y = G(x)$.

So, $y_n = 0$ means $x_n = f(x')$, which means <u>we're on the graph</u>.

We can think of this in terms of the <u>Inverse Function Theorem</u>. What is the Jacobian of G?

$$\mathrm{Jac}(G) = \det \nabla G = \det \begin{pmatrix} & I & 0 \\ & & \vdots \\ -f_{x_1} - f_{x_2} - \cdots - f_{x_{n-1}} & & 1 \end{pmatrix} = 1$$

So, G^{-1} is C^1.

Then, $|\nabla G| \leq C(\partial\Omega)$ and $|\nabla(G^{-1}) \leq C(\partial\Omega)|$.

u is a given function. We can define a new function that has a flat boundary which we can use instead of u.

Define $\tilde{u}(y) := u(G^{-1}(y))$.

Then, $u(x) = \tilde{u}(G(x))$

Suppose $G(\Gamma) = \tilde{G}$.

Then, Step 1 implies,

$$\int_{\tilde{\Gamma}} |\tilde{u}^p| \, \mathrm{d}y' \leq C \int_{\text{shaded region}} [|\tilde{u}|^p + |\nabla \tilde{u}|^p] \, \mathrm{d}y$$

Where the shaded region is $\{x \mid x_n > f(x'), x \in B_R\}$. Now,

[2] $|f|_{C_1} = \sup |f| + \sup |\nabla f|$

$$|\nabla\tilde{u}(y)| \le \left|Du \cdot \nabla(G^{-1})\right| \le C\,|\nabla u|$$

Continuing,

$$\int_{\tilde\Gamma} |\tilde{u}^p| \,\mathrm{d}y' \le C \int_{\text{shaded region}} \left[|\tilde{u}|^p + |\nabla\tilde{u}|^p\right] \mathrm{d}y$$

$$\implies \int_{\Gamma} |u|^p \,\mathrm{d}x' \le C \int_{\Omega\cap B_R} \left[|u|^p + |\nabla u|^p\right] \mathrm{d}x \le C \int_{\Omega} \left[|u|^p + |\nabla u|^p\right] \mathrm{d}x$$

Which finishes Step 2.

3) Decompose $\partial\Omega$ into N_R pieces, and add them up using Step 2.
Since $\partial\Omega$ is compact, $N_R < N(\partial\Omega)$.
[insert picture]
We have: $\forall u \in C^1(\overline{\Omega})$,

$$\int_{\partial\Omega} |u|^p \,\mathrm{d}S \le C(p,\Omega) \int_{\Omega} \left[|u|^p + |\nabla u|^p\right] \mathrm{d}x$$

Now, suppose $u \in W^{1,p}(\Omega)$. Approximate u in $W^{1,p}$ norm by $\{u_m\} \subset W^{1,p}(\Omega) \cap C^1(\overline{\Omega})$. Then,

$$\underbrace{\|u_m - u_l\|_{L^p(\partial\Omega)}}_{\text{cauchy sequence}} \le C\,\|u_m - u_L\|^p_{W^{1,p}}$$

$RHS \to 0$, so $LHS \to 0$ for $m, l \gg 1$.
Therefoe, $\exists Tu \in L^p(\partial\Omega)$ which is the limit of this cauchy sequence.

$\square$

Friday, 9/6/2024

A few comments about traces.
Last time: If $u \in W^{1,p}(\Omega)$ for $1 \le p < \infty$ and $\partial\Omega \in C^1$, we can define the trace of u on $\partial\Omega$, $Tu \in L^p(\partial\Omega)$ and $\exists C = C(p,\Omega)$ such that,

$$\|Tu\|_{L^p(\partial\Omega)} \le C\,\|u\|_{W^{1,p}(\Omega)}$$

Recall: $W_0^{k,p}(\Omega) =$ closure of $C_c^\infty(\Omega)$ in the $W^{k,p}$ norm.
Suppose $k = 1, u \in W_0^{1,p}(\Omega)$. Then, $Tu = 0$ on $\partial\Omega$
Also, if $k > 1$ and $u \in w_0^{k,p}(\Omega)$, we have $Tu = 0$.
In fact, $T(D^\alpha u) = 0 \forall\alpha$ such that $|\alpha| \le k-1$.

Calculus of Variations

Our objects of interest are functionals.
Consider an integral functional of the form:

$$E(u) = \int_\Omega L(x, u, \nabla u)\,\mathrm{d}x$$

We use E because this often denotes Energy. It comes from physics often.
In that case, we call L an energy density. It is also often called the energy density.
A fundamental problem is:
Determine the infimum of E among all functions u in an admissible set $\mathscr{A}$.
We define:

$$m := \inf_{u\in\mathscr{A}} E(u)$$

"First Variation of E"
This is the 'calculus of variations' version of a derivative.
If $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$,

$$L = L(x, z, p)$$

where: x is a point in $\Omega \subset \mathbb{R}^n$
$z \in \mathbb{R}$ [we put u here]
$p \in \mathbb{R}^n$ [we put ∇u here]
We put u in E, but we purtrube it a little bit.
Suppose $u \in \mathscr{A}$, and v such that $u + tv \in \mathscr{A}$ for $|t|$ small.
$E(u + tv)$
Then, $t \mapsto E(u + tv)$ is a function $\mathbb{R} \to \mathbb{R}$. We take derivative and set $t = 0$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u + tv) = \delta E(u; v)$$

Note: if u is a minimum of E, i.e. $E(u) = m$, then the first variation $\delta E(u; v) = 0$ for all v such that $u + tv \in \mathscr{A}$.

Definition 5. u is called a critical point of E in $\mathscr{A}$ if $\delta E(u; v) = 0 \forall v$ such that $u + tv \in \mathscr{A}$.

Question: What is true about a critical point u of an integral functional of form $(*)$?

$$\begin{aligned}
0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u + tv) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_\Omega L(x, u + tv, \nabla u + t\nabla v)\, \mathrm{d}x \\
&= \int_\Omega \left[\frac{\partial L}{\partial z}(x, u, \nabla u)v + \sum_{j=1}^n \frac{\partial L}{\partial p_j}(x, u, \nabla u) v_{x_j} \right] \mathrm{d}x \\
&= \int_\Omega \left[\frac{\partial L}{\partial z}(x, u, \nabla u)v + \nabla_p L \cdot \nabla_x v \right] \mathrm{d}x
\end{aligned}$$

Applying IBP,

$$\begin{aligned}
= \int_\Omega \left[\frac{\partial L}{\partial z}(x, u, Du) - \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial p_j} L(x, u, \nabla u) \right) \right] v \, \mathrm{d}x \\
+ \int_{\partial\Omega} v \nabla_p L(x, u, \nabla u) \cdot \nu \, \mathrm{d}S
\end{aligned}$$

Now, suppose all allowable v's are included in $C_c^\infty(\Omega)$ functions. That would make the boundary term 0. Then, since we can choose v however we want, the big integral is 0.

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial p_j} L(x, u, \nabla u) \right] = \frac{\partial L}{\partial z}(x, u, \nabla u)$$

This is a 2nd order PDE. This is called the Euler-Lagrange equation for E.
Example: Take $E(u) = \displaystyle\int_\Omega \frac{1}{2} |\nabla u|^2 + W(x, u)\, \mathrm{d}x$
Here $L(x, z, p) = \frac{1}{2}|p|^2 + W(x, z)$
Then, we find the Euler-Lagrangian.

$$\frac{\partial L}{\partial p_j}(x, z, p) = \frac{\partial}{\partial p_j} \left[\frac{1}{2}|p|^2 + W(x, u) \right] = p_j$$

$$\frac{\partial L}{\partial p_j}(x, u, \nabla u) = u_{x_j}$$

Also,

$$\frac{\partial L}{\partial z}(x, z, p) = \frac{\partial W}{\partial z}(x, z)$$

So, the Euler-Lagrangian equation gives us:

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(u_{x_j}) = \frac{\partial W}{\partial z}(x, u)$$

$$\Delta u = \frac{\partial W}{\partial z}(x, u)$$

This is a Nonlinear Poisson Equation.
Question: What should we choose $\mathscr{A}$ to be to make our life easiest?
We want integration by parts to be justified, but we want our topology to be weak enough so that finding minimizers is easy.
In the previous example, we have integral of $|\nabla u|^2$. So, ideally we want it to be L^2. So, in the example, best choice is $H^1(\Omega)$.
How does one find a minimzer?
Direct Method: Our problem is, we want to find $u \in \mathscr{A}$ such that:

$$m := \inf_{u \in \mathscr{A}} E(u)$$

Idea: We find a sequence $\{u_j\}$ [called a minimizing sequence]: so that $\{u_j\} \subset \mathscr{A}$ and $E(u_j) \to m$.
Step 1: Try to get a convergent subsequence of $u_{j_k} \to u_* \in \mathscr{A}$. [Compactness]
In this step, if we choose $\mathscr{A}$ to be 'too strong', we lose. Ideally we want $\mathscr{A}$ to be sequentally compact.
Step 2: E might not be continuous in this topology, then we don't have $E(u_*) = m$. So, we want lower semi-continuity.

$$\liminf_{k \to \infty} E(u_{j_k}) \geq E(u_*)$$

Then we have $E(u_*) \leq m$.
This tells us that u_* is a minimum.

Monday, 9/9/2024

Theorem 6 (Extension of Sobolev Functions)**.** Assume $\Omega \subset \mathbb{R}^n$, open , bounded with $\partial\Omega \in C^1$. Then, $\exists C = C(p, \Omega)$ and an extension $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that:

i) $Eu = u$ in Ω

ii) Eu has compact support

iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}$

Idea is, we can let a function down to zero 'gently' and extend that way.

Theorem 7 (Gagliardo - Nirenberg - Sobolev Inequality)**.** Assume $1 \leq p < n$. Then, there exists $C = C(p, n)$ such that for every $u \in C_0^1(\mathbb{R}^n)$ [compactly supported continuously differentiable function] one has the following inequality:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

Where $p^* = \frac{np}{n-p}$ is the critical sobolev exponent.
Note that $\frac{np}{n-p} = \frac{np - p^2 + p^2}{n-p} = p + \frac{p^2}{n-p} > p$.

Question: Where does p^* come from?
Given $u \in C_0^1(\mathbb{R}^n)$, define:

$$u_\lambda(x) \coloneqq u(\lambda x)$$

for $\lambda > 0$.
Suppose $|u|_{L^q(\mathbb{R}^n)} \leq C\,|\nabla u|_{L^p(\mathbb{R}^n)}$
For which q could this be true? Consider the norm of the scaled function:

$$\|u_\lambda\|_{L^q} = \left[\int_{\mathbb{R}^n} |u(\lambda x)|^q \,\mathrm{d}x\right]^{\frac{1}{q}} = \frac{1}{\lambda^{n/q}} \left[\int_{\mathbb{R}^n} |u(y)|^q \,\mathrm{d}y\right]^{\frac{1}{q}}$$

$$\|\nabla u_\lambda\|_{L^p} = \left[\int_{\mathbb{R}^n} |\nabla_x u(\lambda x)|^p \,\mathrm{d}x\right]^{\frac{1}{p}} = \underbrace{\frac{1}{\lambda^{\frac{n}{p}}} (\lambda^p)^{\frac{1}{p}}}_{=\lambda^{1-\frac{n}{p}}} \left[\int_{\mathbb{R}^n} |\nabla_y u(y)|^p \,\mathrm{d}y\right]^{\frac{1}{p}}$$

Applying the same inequality,

$$\frac{1}{\lambda^{\frac{n}{q}}} \left[\int_{\mathbb{R}^n} |u(y)|^q \,\mathrm{d}y\right]^{\frac{1}{q}} \leq C\lambda^{1-\frac{n}{p}} \left[\int_{\mathbb{R}^n} |\nabla u(y)|^p \,\mathrm{d}y\right]^{\frac{1}{p}}$$

$$\|u\|_{L^q} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{q}} \|\nabla u\|_{L^p} \quad \forall \lambda > 0$$

This means $1 - \frac{n}{p} + \frac{n}{q}$ must be 0.
Therefore, $q = p^*$
So, if the theorem is true we must have the q.
Now we prove the theorem.

Proof. First take $n = 3$.
Take $p = 1$.

$$u(x) = \int_{-\infty}^{x_1} u_{x_1}(y_1, x_2, x_3) \,\mathrm{d}y_1$$

$$u(x) = \int_{-\infty}^{x_2} u_{x_2}(x_1, y_2, x_3) \,\mathrm{d}y_2$$

$$u(x) = \int_{-\infty}^{x_3} u_{x_3}(x_1, x_2, y_3) \,\mathrm{d}y_3$$

Therefore,

$$|u(x)| \leq \int_{-\infty}^{x_1} |\nabla u(y_1, x_2, x_3)| \,\mathrm{d}y_1$$

$$|u(x)| \leq \int_{-\infty}^{x_2} |\nabla u(x_1, y_2, x_3)| \,\mathrm{d}y_2$$

$$|u(x)| \leq \int_{-\infty}^{x_3} |\nabla u(x_1, x_2, y_3)| \,\mathrm{d}y_3$$

Multiplying, $|u(x)|^3$

$$\leq \left[\int_{-\infty}^{x_1} |\nabla u(y_1, x_2, x_3)| \,\mathrm{d}y_1\right] \left[\int_{-\infty}^{x_2} |\nabla u(x_1, y_2, x_3)| \,\mathrm{d}y_2\right] \left[\int_{-\infty}^{x_3} |\nabla u(x_1, x_2, y_3)| \,\mathrm{d}y_3\right]$$

Therefore, $|u(x)|^{\frac{3}{2}} \leq$

$$\underbrace{\left[\int_{-\infty}^{\infty} |\nabla u(y_1, x_2, x_3)| \,\mathrm{d}y_1\right]^{\frac{1}{2}}}_{=I_1(x_2,x_3)} \underbrace{\left[\int_{-\infty}^{\infty} |\nabla u(x_1, y_2, x_3)| \,\mathrm{d}y_2\right]^{\frac{1}{2}}}_{=I_2(x_1,x_3)} \underbrace{\left[\int_{-\infty}^{\infty} |\nabla u(x_1, x_2, y_3)| \,\mathrm{d}y_3\right]^{\frac{1}{2}}}_{=I_3(x_1,x_2)}$$

Integrating with respect to x_1 and applying Hölder,

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{3}{2}}\,\mathrm{d}x_1 \leq (I_1(x_2,x_3))^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} I_2(x_1,x_3)\,\mathrm{d}x_1\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} I_3(x_1,x_2)\,\mathrm{d}x_1\right]^{\frac{1}{2}}$$

repeat for x_2 and x_3 and use the fact:

$$\int |f||g| \leq \left[\int f^2\right]^{\frac{1}{2}} \left[\int g^2\right]^{\frac{1}{2}}$$

We see that,

$$\iiint_{\mathbb{R}^3} |u(x)|^{\frac{3}{2}}\,\mathrm{d}x \leq \left[\iiint_{\mathbb{R}^3} |\nabla u(x)|\,\mathrm{d}x\right]^{\frac{3}{2}}$$

Therefore, $\|u\|_{L^{\frac{3}{2}}} \leq \|\nabla u\|_{L^1}$
For $p = 1, n \geq 3$, same proof, but use 'generalized Hölder inequality' given by:

$$\int |u_1 \cdots u_m| \leq \|u_1\|_{L^{p_1}} \cdots \|u_m\|_{L^{p_m}}$$

provided $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$

□

Now, for any $1 < p < n$, let $v(x) = |u(x)|^\gamma$ for some $\gamma > 1$
$v \in C_0^1$ since $u \in C_0^1$. Use previous case [$p = 1$]. Note that, in that case, $p^* = \frac{n}{n-1}$.
Also note that, $\nabla v = \gamma |u|^{\gamma-1}\operatorname{sgn}(u)\nabla u$
Therefore $|\nabla v| \leq \gamma |u|^{\gamma-1}|\nabla u|$
Then,

$$\left[\int |v(x)|^{\frac{n}{n-1}}\right] \leq \int |\nabla v(x)|$$

$$\left[\int |u|^{\frac{\gamma n}{n-1}}\right]^{\frac{n-1}{n}} \leq \gamma \int |u|^{\gamma-1}|\nabla u| \underset{\text{Hölder}}{\leq} \gamma \left[\int |u|^{(\gamma-1)\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \left[\int |\nabla u|^p\right]^{\frac{1}{p}}$$

We pick $\gamma = \frac{p(n-1)}{n-p} > 1$. This gives us,

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} \implies \frac{\gamma n}{n-1} = p^*$$

So we get:

$$\left[\int |u|^{p^*}\right]^{\frac{n-1}{n} - \frac{p-1}{p}} \leq \gamma \|\nabla u\|_{L^p}$$

$$\left[\int |u|^{p^*}\right]^{\frac{n-p}{np}} \leq \gamma \|\nabla u\|_{L^p}$$

$$\|u\|_{L^{p^*}} \leq \gamma \|\nabla u\|_{L^p}$$

Wednesday, 9/11/2024

Theorem 8. Assume $u \in W_0^{1,p}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open, bounded, $\partial\Omega \in C^1$. Then, for $1 \leq p < n$ there exists $C = C(p,n,\Omega)$ such that,

$$\|u\|_{L^{p^*}} \leq C\|\nabla u\|_{L^p(\Omega)}$$

for $p^* = \frac{np}{n-p}$
[Note that before we had condition $u \in C_0^1(\mathbb{R}^n)$, so this is a stronger theorem]

Proof. Extend u to be $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ so that it is compactly supported
[insert picture here]
Then we approximate $\tilde{u}$ by smooth functions $\{u_m\} \subset C_0^\infty(\mathbb{R}^n)$.
Apply previous result to get:

$$\|u_m - u_j\|_{L^{p^*}} \leq C\|\nabla u_m - \nabla u_j\|_{L^p} \to 0$$

Therefore, $\{u_m\}$ is cauchy in $L^{p^*}(\Omega)$.
So we have $u_m \overset{L^{p^*}(\Omega)}{\to} u$
To prove the inequality, we let $m \to \infty$ in the previous result. $\square$

Corollary: For $1 \leq q \leq p^*$ and $1 \leq p < n$, $\exists C = C(p, n, \Omega)$ such that,

$$\|u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$$

Proof. By Hölder,

$$\|v\|_{L^q} \leq C'\|v\|_{L^{p^*}} \quad \forall q < p^*$$

$\square$

This is only half the story since we have $p < n$. What if $p > n$? What about $p = n$?

$p = n$ case

Here $u \in W_0^{1,n}(\Omega)$
Note that $p^* = \frac{np}{n-p}$ which is undefined. So we have problem. For that, we do:
Suppose $\epsilon > 0$. Apply GNS inequality for $p_\epsilon = n - \epsilon$.
Then $p_\epsilon^* = \frac{np_\epsilon}{n-p_\epsilon} = \frac{n(n-\epsilon)}{\epsilon}$. We have:

$$\|u\|_{L^{p_\epsilon^*}(\Omega)} < C_\epsilon\|\nabla u\|_{L^n(\Omega)}$$

Such an inequality exists for every ϵ. So, u is in every $L_q(\Omega)$ space for $1 \leq q < \infty$.
Note that $q = \infty$ not necessary, there are counterexamples.
Corollary[Poincaré Inequality]: For $\Omega \subset \mathbb{R}^n$ open, bounded and for $1 \leq p < \infty$ there exists $C = C(p, n, \Omega)$ such that

$$\|u\|_{L^p(\Omega)} \leq C(p, n, \Omega)\|\nabla u\|_{L^p(\Omega)}$$

This is true for all $u \in W_0^{1,p}(\Omega)$.
This is weaker than previous inequalities, since $p < p^*$.

What if $u \in W^{1,p}$ or $u \in W_0^{1,p}$ and $\Omega \subset \mathbb{R}^n$ and $p > n$?

These functions are even better!
But first we need to talk about Hölder Continuous Functions.

Definition 6 (Hölder Quotient). Let $0 < \alpha < 1$. Define Hölder quotient:

$$[u]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

Definition 7 (Hölder Space). $u \in C^{0,\alpha}(\overline{\Omega})$ if

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} := \sup_{x \in \overline{\Omega}} \Big[|u(x)| + [u]_{C^{0,\alpha}(\overline{\Omega})}\Big] < \infty$$

This is a Banach Space.

Note: if $u \in C^{0,\alpha}(\overline{\Omega})$ then u is uniformly continuous.
Example: Suppose $\Omega = [0,1]$ and $u(x) = x^\beta$ for any $\beta \in (0, \alpha]$.
[graph]
It's derivative goes to ∞ but it is still Hölder continuous.

Theorem 9 (Morrey's Inequality)**.** Assume $n < p \le \infty$. Then $\exists C = C(p,n)$ such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C_0^1(\mathbb{R}^n)$ where $\gamma = 1 - \frac{n}{p}$.

We can approximate so this is also true for $u \in W_0^{1,p}(\Omega)$.
Thus, when $u \in W_0^{1,p}(\Omega)$ we can say that u is in fact Hölder continuous.

Proof. We have to prove that both sup and the Hölder quotient are controlled by the Sobolev norm.
Step 1: Claim: There exists $C = C(n)$ such that $\forall x \in \mathbb{R}^n$ such that $r > 0$,

$$\fint_{B(x,r)} |u(y) - u(x)| \,\mathrm{d}y \le C \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \,\mathrm{d}y \quad \forall u \in C^1(\mathbb{R}^n)$$

Proof of Claim: Fix x. Fix r. Fix $w \in \partial B(0,1)$. Then,

$$|u(x+sw) - u(x)| = \left| \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} u(x+tw) \,\mathrm{d}t \right| \le \int_0^s |\nabla_x u(x+tw)| \cdot \underbrace{|w|}_{=1} \,\mathrm{d}t$$

Integrating over all $w \in \partial B(0,1)$ we get

$$\begin{aligned} \int_{w \in \partial B(0,1)} |u(x+sw) - u(x)| \,\mathrm{d}S_w &\le \int_{w \in \partial B(0,1)} \int_0^s |\nabla_x u(x+tw)| \,\mathrm{d}t \,\mathrm{d}S_w \\ &= \int_0^s \int_{w \in \partial B(0,1)} |\nabla_x u(x+tw)| \underbrace{t^{n-1} \,\mathrm{d}S_w}_{=\mathrm{d}S \text{ on } \partial B(0,t)} \,\mathrm{d}t \end{aligned}$$

Let $y = x + tw \implies t = |y - x|$

$$\begin{aligned} &= \int_{B(x,s)} \frac{|\nabla_y u(y)|}{|y-x|^{n-1}} \,\mathrm{d}y \\ &\le \int_{B(x,r)} \frac{|\nabla_y u(y)|}{|y-x|^{n-1}} \,\mathrm{d}y \end{aligned}$$

TBC

Friday, 9/13/2024

Continuing Morrey's Inequality.
We had:

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)| \,\mathrm{d}S_w \le \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \,\mathrm{d}y$$

Multiplying by s^{n-1},

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)| s^{n-1} \,\mathrm{d}S_w \le \int_{B(x,r)} \frac{|\nabla u(y)| s^{n-1}}{|y-x|^{n-1}} \,\mathrm{d}y$$

Integrating over s from 0 to r,

$$\int_0^r \int_{\partial B(0,1)} |u(x+sw) - u(x)| s^{n-1} \,\mathrm{d}S_w \,\mathrm{d}s \le \int_0^r \int_{B(x,r)} \frac{|\nabla u(y)| s^{n-1}}{|y-x|^{n-1}} \,\mathrm{d}y \,\mathrm{d}s$$

$$\int_0^r \int_{\partial B(x,s)} |u(y)-u(x)| \,\mathrm{d}S_y \,\mathrm{d}s \leq \int_0^r \int_{B(x,r)} \frac{|\nabla u(y)| s^{n-1}}{|y-x|^{n-1}} \,\mathrm{d}y \,\mathrm{d}s$$

$$\int_{B(x,r)} |u(y)-y(x)| \,\mathrm{d}y \leq \frac{r^n}{n} \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \,\mathrm{d}y$$

$$\frac{1}{\alpha(n) r^n} \int_{B(x,r)} |u(y)-y(x)| \,\mathrm{d}y \leq \frac{1}{n\alpha(n)} \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \,\mathrm{d}y$$

which is the claim.
Step 2: (Bounding $\sup|u|$)
Fix $x \in \mathbb{R}^n$.

$$|u(x)| = \fint_{B(x,1)} |u(x)| \,\mathrm{d}y$$

$$\leq \fint_{B(x,1)} |u(x)-u(y)| \,\mathrm{d}y + \fint_{B(x,1)} |u(y)| \,\mathrm{d}y$$

First term: apply step 1. Second term: apply Young's inequality

$$|u(x)| \leq C \int_{B(x,1)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \,\mathrm{d}y + C \left(\int 1^{\frac{p}{p-1}} \,\mathrm{d}y \right)^{\frac{p-1}{p}} \left(\int_{B(x,1)} |u(y)|^p \right)^{\frac{1}{p}} \quad (*)$$

Now apply Hölder's inequality on the first part:

$$\int_{B(x,1)} \frac{1}{|x-y|^{n-1}} |\nabla u(y)| \,\mathrm{d}y \leq \left[\int_{B(x,1)} \left[\frac{1}{|x-y|^{n-1}} \right]^{\frac{p}{p-1}} \,\mathrm{d}y \right]^{\frac{p-1}{p}}$$

We simplify:

$$\left[\int_{B(x,R)} \left[\frac{1}{|y-x|^{n-1}} \right]^{\frac{p}{p-1}} \,\mathrm{d}y \right]^{\frac{p-1}{p}} = \left[\int_0^R \int_{\partial B(0,s)} s^{(n-1)\frac{p}{p-1}} \,\mathrm{d}S \,\mathrm{d}s \right]^{\frac{p-1}{p}}$$

$$= \left[\int_0^R s^{(n-1)\frac{p}{p-1}} \omega(n) s^{n-1} \,\mathrm{d}s \right]^{\frac{p-1}{p}} = C \left[\int_0^R s^{n-1-(n-1)\frac{p}{p-1}} \,\mathrm{d}s \right]^{\frac{p-1}{p}}$$

$$= C \left[R^{n-(n-1)\frac{p}{p-1}} \right]^{\frac{p-1}{p}} = C R^{n\frac{p-1}{p}-(n-1)} = C R^{1-\frac{n}{p}} \quad (**)$$

Applying $R=1$ on $(**)$ and substituting to $()*)$,
$|u(x)| \leq C \left\| \nabla u \right\|_{L^p(\mathbb{R}^n)}$
Step 3: Bounding the Hölder quotient $[u]_{C^{0,\gamma}}$:
Fix x, y in $\mathbb{R}^n$ and let $r = |x-y|$.
Define $W = B(x,r) \cap B(y,r)$.
[insert picture]

$$|u(x)-u(y)| = \fint_{z \in W} |u(x)-u(y)| \,\mathrm{d}z$$

$$\leq \fint_W |u(x)-u(z)| \,\mathrm{d}z + \fint_W |u(y)-u(z)| \,\mathrm{d}z$$

Now,

$$\fint |u(x)-u(z)| \,\mathrm{d}z \leq \frac{1}{|W|} \int_{B(x,r)} |u(x)-u(z)| \,\mathrm{d}z$$

$$\leq C(n) \fint_{B(x,r)} |u(x)-u(z)| \,\mathrm{d}z$$

Applying the claim,

$$\leq C \int_{B(x,r)} \frac{|\nabla u(z)|}{|x-z|^{n-1}} \, \mathrm{d}z$$

Applying Hölder and $(*)$,

$$\leq C \, \|\nabla u\|_{L^p(\mathbb{R}^n)} \, r^{1-\frac{n}{p}}$$

So we're done.

□

Theorem 10. Let $\Omega \subset \mathbb{R}^n$ be bounded, open with $\partial\Omega \in C^1$. Assume $n < p < \infty$. Then, $\exists C(n, p, \Omega)$ such that: for every $u \in W^{1,p}(\Omega)$ one has $u \in C^{0,\gamma}(\Omega)$ where $\gamma = 1 - \frac{n}{p}$ and:

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C \, \|u\|_{W^{1,p}(\Omega)}$$

So, if our Sobolev L^p space is better than our dimension, our function must be continuous, or even better, Hölder Continuous!
Actually, Sobolev Functions are defined upto a set of measure 0. So, we actually have a continuous representative.

Proof. Extension Theorem □

If we have $W^{k,p}$ we can have better statements.
Morrey and G-N-S inequalities can be concatenated.
[See Evans]
Example: Suppose $u \in W^{2,2}(\Omega)$. Suppose $\Omega \subset \mathbb{R}^3$ and bounded.
$u \in H^2(\Omega)$ therefore $u_{x_j} \in W^{1,2}(\Omega)$.
$p = 2, n = 3$. Use GNS. $2^* = \frac{3\times 2}{3-2} = 6$.
Thus, $u_{x_j} \in L^{2^*} = L^6$.
Since also $u \in W^{1,2}$, $u \in L^6$.
Therefore, $u \in W^{1,6}$. We have jumped above the dimension, since $p = 6, n = 3$.
$n < p$ so we can use Morrey.
$\implies u \in C^{0,\gamma}(\Omega)$ where $\gamma = 1 - \frac{n}{p} = 1 - \frac{3}{6} = \frac{1}{2}$.
So, $u \in W^{2,2} \implies u \in C^{0,\frac{1}{2}}(\Omega)$.

Monday, 9/16/2024

Definition 8 (continuously / compactly embedded)**.** Given two Banach spaces X, Y with $X \subseteq Y$ we say X is continuously embedded in Y if $\exists C > 0$ such that:

$$\|x\|_Y \leq C\|x\|_X \forall x \in X$$

X is compactly embedded in Y if it is continuously embedded and every bounded sequence in X is pre-compact in Y.
In other words, if every bounded sequence in X has a Y-convergent subsequence.
Typically people use $X \subset\subset Y$ to denote this.

Theorem 11 (Rellich-Kondrachov Compactness)**.** Let $\Omega \subset \mathbb{R}^n$ open, bounded with $\partial\Omega \subset C^1$. If $1 \leq p < n$ then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for all $q \in [1, p^*)$
[here $p^* = \frac{np}{n-p}$]

Remark: We already have continuous embedding for $q \in [1, p^*]$ by GNS inequality.

Theorem 12 (Hölder Generalization)**.** Assume $1 \leq s < r < t \leq \infty$ and $1 = \frac{r\theta}{s} + \frac{r(1-\theta)}{t}$ for some $\theta \in (0, 1)$.
Then, if $u \in L^s(\Omega) \cap L^t(\Omega)$ then $u \in L^r(\Omega)$ and:

$$\|u\|_{L^r} \leq \|u\|_{L^s}^{\theta} \|u\|_{L^t}^{1-\theta}$$

Proof.

$$\int_\Omega |u|^r = \int_\Omega |u|^{r\theta}|u|^{(1-\theta)r} \leq \left[\int_\Omega |u|^{r\theta\frac{s}{r\theta}}\right]^{\frac{r\theta}{s}} \left[\int_\Omega |u|^{(1-\theta)r\frac{t}{r(1-\theta)}}\right]^{\frac{r(1-\theta)}{t}}$$

$$= \left[\int_\Omega |u|^s\right]^{\frac{r\theta}{s}} \left[\int_\Omega |u|^t\right]^{\frac{r(1-\theta)}{t}}$$

□

Proof of Rellich-Kondrachov. Assume $\|u_m\|_{W^{1,p}(\Omega)} < C_0$
Step 1: Mollify $u_m \rightsquigarrow u_m^\epsilon$ and argue that u_m^ϵ approximates u_m [uniformly in m] in L^q.
Step 2: Apply Arzela-Ascoli to $\{u_m^\epsilon\}$ for each ϵ fixed.
Step 3: Use diagonalization argument. We control things when m is fixed, we control things when ϵ is fixed so we can vary both.
First extend u_m to be compactly supported in $V' \supset \Omega$. Still have $\|u_m\|_{W^{1,p}(V')} < \tilde{C}_0$.
Now Mollify: $u_m^\epsilon := \eta_\epsilon * u_m$. $\eta_\epsilon(x) = \epsilon^{-n}\eta\left(\frac{x}{\epsilon}\right)$ with $\eta \in C_0^\infty$ and $\text{supp}(\eta) \subset B(0,1)$ with $\int \eta = 1$
Note u_m^ϵ is compactly supported. Say $\text{supp}(u_m^\epsilon) \subset V$. $V' \subset V$.
Claim: $\{u_m^\epsilon\}$ converges as $\epsilon \to 0$ to u_m in $L^q(\Omega)$ uniformly in m. That is, we claim: $\forall \delta > 0$ there exists $\epsilon_0 > 0$ such that $\forall \epsilon < \epsilon_0$

$$\|u_m^\epsilon - u_m\|_{L^q(V)} < \delta$$

$\forall m$.
To see this, first assume u_m is smooth.

$$u_m^\epsilon(x) - u_m(x) = \int_{B(0,\epsilon)} \eta_\epsilon(y)\left(u_m(x-y) - u_m(x)\right) \mathrm{d}y$$

Let $z = \frac{y}{\epsilon}$

$$= \int_{B(0,1)} \eta(z)\left[u_m(x-\epsilon z) - u_m(x)\right] \mathrm{d}z$$

$$= \int_{B(0,1)} \eta(z) \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u_m(x-\epsilon tz)\,\mathrm{d}t\,\mathrm{d}z$$

$$= -\epsilon \int_{B(0,1)} \eta(z) \int_0^1 \nabla u_m(x-\epsilon tz)\cdot z\,\mathrm{d}t\,\mathrm{d}z$$

$$\leq \epsilon \int_V \int_{B(0,1)} \int_0^1 \eta(z)|\nabla u_m(x-\epsilon tz)|\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x$$

Let $y' = x - \epsilon tz$

$$= \epsilon \int_{\tilde{V}} \int_{B(0,1)} \int_0^1 \eta(z)|u_m(y')|\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}y'$$

$$= \epsilon \int_{\tilde{V}} |\nabla u_m(y')|\,\mathrm{d}y'$$

Apply Hölder:
$\leq \epsilon C\|\nabla u_m\|_{L^p} \leq \epsilon\tilde{\tilde{C}}_0$
If u_m is not smooth then approximate u_m by $\tilde{u}_m$ smooth such that $\|\tilde{u}_m - u_m\|_{W^{1,p}} < \delta$ for all m.Then,

$$\int_V |u_m^\epsilon(x) - u_m(x)|\,\mathrm{d}x \leq$$

$$\int_V |u_m^\epsilon(x) - \tilde{u}_m(x)|\,\mathrm{d}x + \int_V |\tilde{u}_m(x) - u_m(x)|\,\mathrm{d}x < \delta + \delta$$

Now apply interpolation:

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - \tilde{u}_m\|_{L^1}^\theta \|\tilde{u}_m - u_m\|_{L^{p^*}}^{1-\theta}$$

Here $s = 1, t = p^*$
$\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$
Recall GNS gave us $\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \tilde{C}_0$

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$$|u_m^\epsilon| \leq \int_V \epsilon^{-n} \eta\left(\frac{x-y}{\epsilon}\right) |u_m(y)| \,\mathrm{d}y$$

$$\leq \epsilon^{-n} \int_V |u_m(y)| \,\mathrm{d}y$$

$$\leq \epsilon^{-n} \|u_m\|_{W^{1,p}(V)} < \text{constant}$$

Note that

$$\nabla u_m^\epsilon(x) = \int_V \epsilon^{-n} \nabla\eta\left(\frac{x-y}{\epsilon}\right) \frac{1}{\epsilon} u_m(y) \,\mathrm{d}y$$

$$|\nabla u_m^\epsilon(x)| \leq \int_V \epsilon^{-n} \left|\nabla\eta\left(\frac{x-y}{\epsilon}\right)\right| \frac{1}{\epsilon} |u_m(y)| \,\mathrm{d}y$$

$$\leq \epsilon^{-(n+1)} \sup |\nabla\eta| \int |u_m(y)| \,\mathrm{d}y$$

$$\leq \epsilon^{n+1} \sup |\nabla\eta| \|u_m\|_{W^{1,p}(V)} \leq \text{constant}$$

Therefore, u_m^ϵ is equi-Lipschitz and thus equicontinuous.
Step 3:
Claim: We can find a subsequence $\{u_{m_j}\}$ of the u_m so that:

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} < \delta$$

Use step 1 to find ϵ such that $\|u_m^\epsilon - u_m\|_{L^q(V)} < \frac{\delta}{2}$ for all m.
Apply Arzela-Ascoli for that ϵ to get subsequence u_{m_j} such that:

$$\limsup_{j,k\to\infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\| \to 0$$

$$\implies \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} = 0$$

Therefore,

$$\left\|u_{m_j} - u_{m_k}\right\|_{L^q(V)} \leq \left\|u_{m_j} - u_{m_j}^\epsilon\right\|_{L^q} + \left\|u_{m_j}^\epsilon - u_{m_k}^\epsilon\right\|_{L^q} + \left\|u_{m_k}^\epsilon - u_{m_k}\right\|_{L^q}$$

Take $\limsup_{j,k\to\infty}$ to see $< \frac{\delta}{2} + 0 + \frac{\delta}{2} = \delta$
Then take $\delta = 1, \frac{1}{2}, \frac{1}{4} \cdots$ [subsequences of subsequences] to get $u_{m_{l_k}} \xrightarrow{L^q} \overline{u}$.

$\square$

Remark: What if $\|u_m\|_{W^{1,p}(\Omega)} < C_0$ with $n < p$?
Morrey's inequality gives us:

$$\|u_m\|_{C^{0,\alpha}} \leq \text{constant}$$

$$|u_m(x) - u_m(y)| \leq C|x-y|^\alpha$$

Thus we also have equicontiunity.

Elliptic PDE Definition

Suppose $u =$ density, amount/volume, $u = u(x,t), x \in \mathbb{R}^3$.
Let $\Omega \subset \mathbb{R}^3$ be any domain where the whole process is taking place.
Suppose we want to know the amount of stuff in Ω in time t. It is:

$$\int_\Omega u(x,t)\,\mathrm{d}x$$

Now we ask: how does it change w.r.t. time?

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)\,\mathrm{d}x$$

This depends on the stuff enterring or exiting [= flux accross $\partial\Omega$] + sources/sinks.
What is flux? It must be a vector $\vec{Q}(x,t)$. So, total contribution of flux is:

$$-\int_{\partial\Omega} \vec{Q}(x,t)\cdot\nu\,\mathrm{d}S$$

Plus sources/sinks density.

$$+\int_\Omega F(x,t)\,\mathrm{d}x$$

Thus we have,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_\Omega u(x,t)\,\mathrm{d}x = -\int_{\partial\Omega} \vec{Q}(x,t)\cdot\nu\,\mathrm{d}S + \int_\Omega F(x,t)\,\mathrm{d}t$$

The choice of flux $\vec{Q}$ distinguishes different physical settings.
Let's rewite our equation as 1 volume integral.

$$\int_\Omega \left[u_t(x,t) + \operatorname{div}\vec{Q}(x,t) - F(x,t)\right]\,\mathrm{d}x = 0$$

This is true for arbitrary Ω so we have:

$$u_t = -\operatorname{div}(\vec{Q}) + F$$

Naturally, we pick Q to model a diffusion process.
If there's a lot of stuff 'inside' then stuff will go outside and vice versa. So, we can take,

$$\vec{Q} = -k\nabla u$$

Or more generally,

$$\vec{Q} = -A(x)\nabla u$$

where A is positive definite 3×3 matrix.
For $Q = -k\nabla u$ we have:

$$u_t = \operatorname{div}(k\nabla u) + F = k\Delta u + F$$

which is the heat equation.
For $Q = -A(x)\nabla u$,

$$u_t = \operatorname{div}(A(x)\nabla u) + F$$

This is a model where stuff 'smooths out / settles down'.
As $t\to\infty$ we expect $u(x,t)\to\tilde{u}(x)$.
In that case, $-\operatorname{div}(A(x)\nabla u) = F$.
If A has entries $a_{ij}(x)$, then,

$$\operatorname{div}(A(x)\nabla u) = \sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}(x)u_{x_j})$$

This is an elliptic operator [if $a_{ij} = a_{ji}, A$ positive definite].
This is the divergence form.

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Assume A is symmetric. We define the linear elliptic operator L by:

i) $Lu = -\frac{\partial}{\partial x_j}(a_{ij}u_{x_i}) + b_i(x)u_{x_i} + c(x)u$ [divergence form elliptic operator]

ii) $Lu = -a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} + c(x)u$ [non divergence form] where $b_i : \Omega \to \mathbb{R}, i = 1, \cdots, n, c : \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}^n$. If a_{ij} are C^1 then $a_{ij}(x)u_{x_ix_j} = \frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) - \left(\frac{\partial}{\partial x_j}a_{ij}(x)\right)u_{x_i}$

Why elliptic? $A = (a_{ij})$ and $a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall \xi \in \mathbb{R}^n$ for some $\theta > 0$.

Definition 9. We say $u \in H_0^1(\Omega)$ [same as $W_0^{1,2}$] is a weak solution to the equation $Lu = f$ for some $f \in L^2(\Omega)$ and $u = 0$ on $\partial\Omega$ [homogeneous Dirichlet condition] if:

$$\int_\Omega A\nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega b_i(x)u_{x_i}v \,\mathrm{d}x + \int_\Omega c(x)uv \,\mathrm{d}x = \int_\Omega vf \,\mathrm{d}x \quad (*)$$

for all $v \in H_0^1(\Omega)$.

Definition 10. We say $u \in H^1(\Omega)$ is a weak solution to

$$\begin{cases} Lu = f, & \text{in } \Omega; \\ A\nabla u \cdot \nu_{\partial\Omega} = 0, & \text{on } \partial\Omega \end{cases}$$

[Neumann boundary condition]
if $(*)$ holds for all $v \in H^1(\Omega)$.

Suppose

$$\int_\Omega a_{ij}(x)u_{x_i}v_j + b_i(x)u_{x_i}v + cuv \,\mathrm{d}x + \int_{\partial\Omega} vA\nabla u \cdot \nu \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x \quad (**)$$

why? Suppose u was a classical solution. Then integrating by part yields the second equation.

Functional Analysis

Background on Hilbert Spaces
Recall: a Hilbert Space H is a Banach space [normed and complete] that posses an inner product $(\cdot,\cdot)_H$ such that $\|\cdot\|$ is inherited from the inner product.
Basically: complete space with inner product.
Example: $\mathbb{R}^n$ with dot product, $L^2(\Omega)$ with $(u,v)_{L^2} = \int_\Omega uv \,\mathrm{d}x$
Or, $H^1(\Omega)$ with $(u,v)_H = \int_\Omega uv \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$
Or, most importantly, in $H_0^1(\Omega)$ we have $(u,v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$
By Poincare, then $C_1\|u\|_{H^1}^2 \leq \|u\|_{H_0^1}^2 = \int_\Omega |\nabla u|^2 \,\mathrm{d}x \leq C_2\|u\|^2$
Given $u, v \in H$ we'll say u is orthogonal to v if $(u,v) = 0$.
Given a subspace $M \subset H$, we'll write $M^\perp := \{v \in H : (u,v) = 0 \forall u \in M\}$

Proposition 1. Let $M \subset H$ be a closed subspace of H. Then $\forall x \in H, \exists y \in M, z \in M^\perp$ such that $x = y + z$.

Proof. Idea: y is the point closest to x in M.
Consider $x \notin M$. Define

$$d = \inf_{x' \in M} \|x - x'\|$$

Then there must be a sequence $\{y_n\} \subset M$ such that $\|x - y_n\| \to d$.
Recall:

$$\|X - Y\|^2 + \|X + Y\|^2 = 2\|X\|^2 + 2\|Y\|^2$$

Pick $X = x - y_n, Y = x - y_m$. Then,

$$\|y_m - y_n\|^2 + 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2$$

Therefore,

$$4d^2 + \|y_n - y_m\|^2 \le 2\|x - y_n\|^2 + 2\|x - y_m\|^2$$

$$\|y_n - y_m\|^2 \le 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4d^2$$

Taking limsup we see that $\{y_n\}$ is cauchy.
Define $z = x - y$. It is easy to see that $z \in M^\perp$.

$\square$

Monday, 9/23/2024

Divergence from Linear Elliptic Operator

$$L(u) = -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u$$

Uniform ellipticity:
$a_{ij} = a_{ji}$ symmetric
$a_{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2 \forall \xi \in \mathbb{R}^n \forall x \in \Omega$ for some $\theta > 0$.
Meaning minimum eigenvalue is uniformly bigger than some θ
Review: $u \in H_0^1(\Omega)$ is a weak solution to $Lu = f$ in Ω for $f \in L^2(\Omega)$ and $u = 0$ on $\partial\Omega$ if:

$$B(u, v) := \int_\Omega \left[a_{ij}(x)u_{x_i}v_{x_j} + b_i(x)u_{x_i}v + c(x)uv\right] \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x \,\forall v \in H_0^1(\Omega)$$

u is a weak solution to $Lu = f$ in Ω and $A\nabla u \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$ if

$$B(u, v) = \int_\Omega fv \,\mathrm{d}x \,\forall v \in H^1(\Omega)$$

Note that B is a bilinear form.

Definition 11. A bounded linear operator $L : X \to Y$ that is <u>linear</u> that satisfies:

$$\|L\| := \sup_{\|x\|_X \le 1} \|L(x)\|_Y < \infty$$

For linear operators, boundedness is the same as continuity.

Definition 12. A bounded linear functional on X is a bounded linear operator $L : X \to \mathbb{R}$

<u>Notation</u>: If u^* is a bounded linear functional we'll often write $\langle u^*, x\rangle$ for the evaluation of u^* at x.

$$\|u^*\| = \sup_{\|x\| \le 1} \langle u^*, x\rangle$$

Definition 13. The dual of a Banach Space X is the set of bounded linear functionals on X. Notation: X^*

Theorem 13 (Riesz Representation Theorem)**.** Assume H is a Hilbert space with inner product $(\cdot,\cdot)$.
For every bounded linear functional $u^* : H \to \mathbb{R}$ there exists a unique $u \in H$ such that $(u^*, v) = (u, v) \forall v \in H$ and $\|u^*\| = \|u\|$

Proof. Suppose $u^* \in H^*$.
Let N be the nullspace of u^*:

$$N = \{v \in H : \langle u^*, v\rangle = 0\}$$

If $N = H$ then $\langle u^*, -\rangle = 0$ so u must be 0. Assume N is a proper subspace. N must be closed since u^* is continuous.
Fix $z \in N^\perp$, $\langle u^*, z\rangle \neq 0$.
Then $\forall x \in H$ we have:

$$\left\langle u^*, x - \frac{\langle u^*, x\rangle}{\langle u^*, z\rangle} z \right\rangle = \langle u^*, x\rangle - \langle u^*, x\rangle = 0$$

Thus, $x - \frac{\langle u^*, x\rangle}{\langle u^*, z\rangle} \in N$.

$$\left(x - \frac{\langle u^*, x\rangle}{\langle u^*, z\rangle} z, z \right) = 0$$

$$\implies (x, z) = \frac{\langle u^*, x\rangle}{\langle u^*, z\rangle} \|z\|^2$$

$$\implies \langle u^*, x\rangle = \left(\frac{\langle u^*, z\rangle}{\|z\|^2} z, x \right)$$

We just take $u := \frac{\langle u^*, z\rangle}{\|z\|^2} z$
Uniqueness: if $\langle u^*, x\rangle = (u_1, x) = (u_2, x)$ then $(u_1 - u_2, x) = 0$, choose $x = u_1 - u_2$ to deduce that $u_1 = u_2$.

$$\|u^*\| = \sup_{\|x\| \leq 1} \langle u^*, x\rangle = \sup_{\|x\| \leq 1} (u, x) \leq \sup_{\|x\| \leq 1} \|u\|_H \|x\|_H \leq \|u\|_H$$

On the other hand,

$$\|u\| = \frac{(u, u)}{\|u\|} = \left\langle u^*, \frac{u}{\|u\|} \right\rangle \leq \|u^*\|$$

Thus $\|u\| = \|u^*\|$

□

Solving a PDE [finally]

Poisson: Find a weak solution to $-\Delta u = f$ where $f \in L^2(\Omega)$ in Ω and $u = 0$ on $\partial\Omega$
We seek $u \in H_0^1(\Omega)$ satisfying

$$B(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \, \forall v \in H_0^1(\Omega)$$

Recall: By Poincaré's inequality, an inner product for H_0^1 can be taken as:

$$(u, v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$$

We seek u such that:

$$(u, v)_{H_0^1} = \int_\Omega f v \, \mathrm{d}x \, \forall v \in H_0^1(\Omega)$$

Existence of unique u follows from Riesz Representation. We just need to show that $v \mapsto \int_\Omega fv \, dx$ is a bounded linear functional on $H_0^1(\Omega)$.
It is obviously linear.

$$\sup_{\|v\|_{H_0^1} \leq 1} \int_\Omega fv \, dx \leq \sup_{\|v\|_{H_0^1} \leq 1} \|f\|_{L^2} \|v\|_{L^2} \leq c_p \cdot 1 \|f\|_{L^2}$$

Where c_p is the Poincaré constant: $\int u^2 \leq c_p \int |\nabla u|^2 \, \forall u \in H_0^1$

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Last time, we used Riesz Representation Theorem to get existene of a weak solution to $Lu = f$ in Ω and $u = 0$ on $\partial\Omega$
for $Lu = -\frac{\partial}{\partial x_j}(a_{ij}(x) u_{x_i}) = f$
a_{ij} elliptic.
Weak formulation: seek $u \in H_0^1(\Omega)$ such that:

$$B[u, v] := \int_\Omega a_{ij} u_{x_i} v_{x_j} \, dx = \int_\Omega fv \, dx \quad \forall v \in H_0^1(\Omega)$$

For $-\Delta u = f$ we seek u such that $B[u, v] = \int_\Omega \nabla u \cdot \nabla v$ where $B[u, v] = (u, v)_{H_0^1(\Omega)}$
How do we obtain a weak solution for

$$Lu := -\frac{\partial}{\partial x_j} (a_{ij}(x) u_{x_i}) + b_i(x) u_{x_i} + c(x) u = f$$

Consider:

$$B[u, v] = \int_\Omega a_{ij} u_{x_i} v_{x_j} + b_i(x) u_{x_i} v + c(x) uv \, dx$$

Not symmetric so not an inner product but that's not the only problem. Consider 1-dim.

$$-u'' - u = 1 \quad \Omega = (0, \pi)$$

$u(0) = u(\pi) = 0$.
Suppose u solves this.
Multiply the ODE by $\sin x$ and integrate.

$$\int_0^\pi -\sin x \cdot u'' - \sin x \cdot u \, dx = \int_0^\pi \sin x \, dx = 2$$

Note

$$\int_0^\pi -\sin x \cdot u'' \, dx = [-\sin x \cdot u']_0^\pi - \int_0^\pi \cos x \cdot u' \, dx = [-\cos x \cdot u]_0^\pi + \int_0^\pi \sin x \cdot u \, dx$$

Therefore, by plugging it into the original argument,

$$0 = 2$$

Thus we don't have solutions!
Today: Lax-Milgram Lemma

Theorem 14 (Lax-Milgram Lemma)**.** Assume $B : H \times H \to \mathbb{R}$ is a bilinear form on a hilbert space H. Suppose,

i) $\exists \alpha > 0$ such that $B(u, v) \leq \alpha \|u\| \cdot \|v\| \forall u, v \in H$

ii) $\exists \beta > 0$ such that $B[u, u] \geq \beta \|u\|^2$

Then we have the same conclusion as Riesz: $\forall f \in H^* \exists! u \in H$ such that $B[u, v] = \langle f, v \rangle \forall v \in H$

Proof. For any fixed $u \in H$ consider the map $v \mapsto B[u, v]$.
It is a bounded linear functional in H.
Apply Riesz Representation: for each fixed u there exists unique $w \in H$ such that $B[u, v] = (w, v) \forall v$.
We write $Au = w$
Then $B[u, v] = (Au, v)$
Claim: $A : H \to H$ is linear, bounded, 1-1
Linearity: $B[c_1u_1 + c_2u_2, v] = (A(c_1u_1 + c_2u_2), v)$
$= c_1B[u_1, v] + c_2B[u_2, v] = c_1A(u_1, v) + c_2A(u_2, v)$.
A is bounded since $\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha\|u\|\|Au\|$
A is 1-1 since $\beta\|u\|^2 \leq B[u, u] = (Au, u) \leq \|Au\|\|u\|$
So $\beta\|u\| \leq \|Au\|$
So, if $u \neq 0$ we have $Au \neq 0$ so 0 is the only element in the kernel, so it is 1-1.
Claim: range of $A, R(A)$ is closed.
Consider $\{w_j\} \subset R(A)$. Suppose $w_j \to w$. $\exists\{u_j\} \subset H$ so that $Au_j = w_j$
Since w_j are cauchy, u_j are cauchy:
$\|w_j - w_k\| = \|Au_j - Au_k\| \geq \beta\|u_j - u_k\|$
$\implies$ u_j are cauchy. So $u_j \to u \in H$ and by continuity $Au_j \to Au = w$ and thus $R(A)$ is closed.
Claim: $R(A) = H$.
If not, apply projection lemma. $\exists w \in R(A) \perp$ so that $w \neq 0$.

$$\beta\|u\|^2 \leq B[w, w] = (Aw, w)$$

Since $Aw \in R(A)$ we have the inner product is 0 and thus $w = 0$.
Now, let $f \in H^*$ be any bounded linear functional. Apply Riesz to show that there exists unique $w \in H$ such that $\langle f, v\rangle = (w, v) = (Au, v) = B[u, v]$ for unique u.

□

Theorem 15. For $Lu = -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u$ with a_{ij} elliptic and with $a_{ij}, b_i, c \in L^\infty(\Omega)$.
Then there exists a number $\gamma > 0$ such that $\forall\mu \geq \gamma$:
a weak $H_0^1(\Omega)$ solution exists to:

$$Lu + \mu u = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

$\forall f \in L^2(\Omega)$

Proof. We just apply Lax-Milgrim
We will prove:

i) $|B[u, v]| \leq \alpha\|u\|_{H_0^1}\|v\|_{H_0^1}$

ii) $B[u, u] + \gamma\|u\|_{L^2}^2 \geq \beta\|u\|_{H_0^1}^2$

First Condition: Need to check $B[u, v] \leq \alpha\|u\|_{H_0^1}\|v\|_{H_0^1}$ for some α

$$B[u, v] = \int_\Omega a_{ij}(x)u_{x_i}v_{x_j}b_i(x)u_{x_i}v + c(x)uv \,\mathrm{d}x$$

$$\leq B[u, v] \leq \int_\Omega n^2M|\nabla u||\nabla v| + nM|\nabla u||v| + M|u||v| \,\mathrm{d}x$$

$$\leq n^2M\|u\|_{H_0^1}\|v\|_{H_0^1} + nM\|u\|_{H_0^1}\|v\|_{L^2} + M\|u\|_{L^2}\|v\|_{L^2}$$

Apply Poincaré

$$\leq \alpha\|u\|_{H_0^1}\|v\|_{H_0^1}$$

So we have the first condition.

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Now we check the second condition.

$$B[u,u] = \int_\Omega a_{ij}(x)u_{x_i}u_{x_j} + b_i(x)u_{x_i}u + c(x)u^2 \,\mathrm{d}x$$

Use the fact that a_{ij} is elliptic, meaning $a_{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2$

$$\geq \theta \int_\Omega |\nabla u|^2 - nM|\nabla u||u| - M|u|^2 \,\mathrm{d}x$$

We don't have to worry about the $-M|u|^2$ because we can choose γ. We need to deal with $-nM|\nabla u||u|$.
Recall that AM-GM implies:

$$\varepsilon^2 a^2 + \frac{1}{4\varepsilon^2}b^2 \geq ab$$

$$\geq \theta \int_\Omega |\nabla u|^2 - nM\left(\varepsilon|\nabla u|^2 + \frac{1}{4\varepsilon^2}|u|^2\right) - M|u^2| \,\mathrm{d}x$$

$$\geq \theta \int_\Omega (1 - nM\varepsilon)\,|\nabla u|^2 - \left(M + \frac{nM}{4\varepsilon^2}\right)|u|^2 \,\mathrm{d}x$$

We can choose ε so that $nM\varepsilon = \frac{\theta}{2}$ and then choose appropriate γ. So we're done. □

Theorem 16. For L as defined, $\forall \mu \geq \gamma \forall f \in L^2(\Omega)$ there exists a unique weak solution to $L_\mu u = f, u = 0$ on $\partial\Omega$ where $L_\mu \coloneqq L + \mu$

Proof. Lax-Milgrim: $B_\mu[u,v] \coloneqq B[u,v] + \mu \int uv$ so L_μ has inverse. □

Functional Analysis

Definition 14 (Adjoint)**.** Given a bounded linear operator on Hilbert spaces

$$A : H \to H \quad H \text{ is a Hilbert space}$$

The adjoint of A is the bounded linear operator $A^* : H \to H$ defined by:

$$(x, A^*y) = (Ax, y) \quad \forall x, y \in H$$

If $A = A^*$ we say A is self-adjoint.
Example: If $H = \mathbb{R}^n$ then A is a matrix and A^* is the transpose.

Definition 15 (Compact bounded linear operator)**.** Let X, Y be Banach spaces. A bounded linear operator $K : X \to Y$ is compact if for every sequence $\{x_j\} \subset X$ such that $\|x_j\|_X$ is uniformly bounded then there exists a subsequence and $y \in Y$ such that:

$$Kx_{j_l} \to y$$

Example: Suppose $X = C([0,1])$ and $Y = C^1([0,1])$. $\forall f \in X$ define $Kf = u$ provided u solves $u'' = f$ on $(0,1)$ and $u(0) = 0, u'(0) = 0$. We have a formula:

$$u(x) = \int_0^x \int_0^y f(s) \,\mathrm{d}s \,\mathrm{d}y$$

Suppose $\|f_j\|_{C([0,1])} \leq M$. Then,

$$u_j(x) = \int_0^x \int_0^y f_j(s) \,\mathrm{d}s \,\mathrm{d}y$$

$$|u_j| \leq M$$

$$|u_j'| \le \left|\int_0^x f_j(s)\,\mathrm{d}s\right| \le M$$

$$\|u_j''\| = \|f_j\| \le M$$

Apply Arzela Ascoli to find:

$$u_{j_k} \stackrel{C^1}{\to} u_0$$

Thus K must be compact.

Theorem 17 (Fredhome Alternative)**.** Assume $K : H \to H$ is a compact operator on a Hilbert space. Then either:

i) The homogeneous equation $x - Kx = 0$ has a non-trivial solution. OR

ii) $\forall y \in H, \exists! x \in H$ such that $x - Kx = y$

Proof. There are 4 steps.
Step 1: Instead of $x - Kx$ we write $S := I - K$. $N(S)$ denotes the nullspace of S.
Claim: $\exists C$ such that:

$$\mathrm{dist}(x, N(S)) \le C\|Sx\| \,\forall x \in H$$

Proof: Suppose we cannot find such a C. Then we can find a sequence $\{\tilde{x}_k\} \subset H$ such that:

$$\mathrm{dist}(\tilde{x}_k, N(S)) \ge k\|S\tilde{x}_k\|$$

Replace with $x_k = \frac{\tilde{x}_k}{\|S\tilde{x_k}\|}$. Then $\|Sx_k\| = 1$ and

$$d_k := \mathrm{dist}(x_k, N(S)) \to \infty$$

Thus, we can find $\{y_k\} \subset N(S)$ such that:

$$d_k \le \|x_k - y_k\| \le 2d_k$$

define $z_k := \frac{x_k - y_k}{\|x_k - y_k\|}$ and so $\|z_k\| = 1$.
But $\|Sz_k\| = \frac{1}{d_k}\|Sx_k - Sy_k\| = \frac{1}{d_k}\|Sx_k\| = \frac{1}{d_k} \to 0$
Therefore $Sz_k \to 0$
K is compact, so we have subsequence $Kz_{k_j} \to y_0 \in H$.
$S_{z_{k_j}} \to 0 \implies z_{k_j} - Kz_{k_j} \to 0$ and since $K_{z_{k_j}} \to y_0$ we have $z_{k_j} \to y_0$.
Since $Sz_{k_j} = 0$ we have $Sy_0 = 0$ by continuity. Thus $y_0 \in N(S)$
However, $\mathrm{dist}(z_{k_j}, N(S)) = \inf_{y \in N(S)}\|z_{k_j} - y\| = \inf_{y\in N(S)}\left\|\frac{x_{k_j} - y_{k_j}}{\|x_{k_j} - y_{k_j}\|} - y\right\|$
$= \frac{1}{\|x_{k_j} - y_{k_j}\|}\inf_{y \in N(S)}\|x_{k_j} - \underbrace{(y\|x_{k_j} - y_{k_j}\| + y_k)}_{\in N(S)}\|$
$\ge \frac{1}{\|x_{k_j} - y_{k_j}\|} d(x_{k_j}, N(S)) \ge \frac{d_{k_j}}{2d_{k_j}} = \frac{1}{2}$
z_{k_j} converges to something in $N(S)$ but is a set distance away from $N(S)$, which is impossible. Thus we have proved the claim.

Monday, 9/30/2024

Step 2: Claim: Let $R(S) =$ range of S. Then, $R(S)$ is a closed subspae of H.
Proof: Consider a sequence $\{x_k\} \subset H$ so that $Sx_k \to y$ for some y. We must show that $y \in R(S)$.
From step 1, define $d_k := \mathrm{dist}(x_k, N(S)) \le C\|Sx_k\| \to \|y\|$.
d_k is uniformly bounded.
By projection theorem, we can find:

$$d_k = \|\underbrace{x_k - y_k}_{w_k}\| \quad y_k = \text{closest point to } x_k \text{ in } N(S).$$

Then $\|w_k\| \leq \text{const}$.
Since K is compact, $Kw_{k_j} \to w_0 \in H$
Thus, $Sx_{k_j} = x_{k_j} - Kx_{k_j} \to y$
Thus, $x_{k_j} \to y + w_0$
Since S is continuous, $S(y + w_0) = y$
So, $y \in R(S)$.
Step 3: If $N(S) = \{0\}$ then $R(S) = H$.
Let $R_j = \text{range of } S^j [= S(S(\cdots S(H)))]$.
By Step 2, $\{R_j\}$ is a sequence of closed subspaces. Furthermore, it is non-increasing.
We claim that it eventually stabilizes.
Suppose, for contradiction, the sequence keeps decreasing.
By projection theorem, $\forall j \exists y_j \in R_j$ such that $\|y_j\| = 1$ and $\text{dist}(y_j, R_{j+1}) \geq \frac{1}{2}$.
Let $n > m$. We look at $Ky_m - Ky_n$.

$$Ky_m - Ky_n = (I - S)y_m - (I - S)y_n = y_m - (\underbrace{Sy_m}_{\in R_{m+1}} - \underbrace{Sy_n}_{\in R_{n+1} \subset R_{m+1}} + \underbrace{y_n}_{\in R_n \subset R_{m+1}})$$

Thus, $\|Ky_m - Ky_n\| \geq \frac{1}{2}$.
This contradicts the compactness of K.
Therefore, $\exists k$ such that $R_j = R_k \forall j > k$.
So, assume $N(S) = \{0\}$. Let $y \in H$.
We have $S^k y \in R_k = R_{k+1}$.
Therefore, $S^k y = S^{k+1} x$ for some x.
Therefore, $S^k(y - Sx) = 0$.
Since $N(S) = \{0\}$ we have $y = Sx$.
Step 4. If $R(S) = H$ then $N(S) = \{0\}$.
Let $N_j :=$ nullspace of S^j. Now, N_j is a non-decreasing sequence of closed subspaces.
Claim: $\exists k$ such that $N_j - N_k \forall j > k$. Argue by contradiction as before.
Assume $R(S) = H$.
$\forall y \in N_k, S^k y = 0$ and furthermore $\exists x_1$ such that $y = Sx_1$. Repeating, $y = S^k x$.
So, $S^{2k} x = 0$. Since null space stabilzes after k we have $S^k x = 0$. Therefore $y = 0$.
Thus $N_k = \{0\} \implies N(S) = \{0\}$.
In case ii why is $(I - K)^{-1}$ bounded?
Use Step 1: $\text{dist}(x, N(S)) \leq C\|Sx\| = C\|(I - K)x\|$
For ii we have $N(S) = \{0\}$ so:

$$\|x\| \leq C\|Sx\|$$

Writing $Sx = y$,

$$\|(I - K)^{-1} y\| \leq C\|y\|$$

□

Given a bounded linear operator $T : H \to H$ [could be a normed linear space as well],

Definition 16. The resolvent set of T is:

$$\rho(T) = \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijective}\}$$

The spectrum of T is:

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

We can substitute real for complex.

Definition 17. $\lambda \in \sigma(T)$ is called an eigenvalue of T if $\exists x \in H$ such that $Tx = \lambda x$.

Note: $(\sigma(T) \setminus \{\text{eigenvalues}\})$ is called the continuous spectrum.

Wednesday, 10/2/2024

$$Lu := -(a^{ij}u_{x_i})_{x_j} + b^i(x)u_{x_i} + c(x)u$$

$$a^{ij} = a^{ji}, a^{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2, \theta > 0$$

$$a^{ij}, b^i, c \in L^\infty$$

Recall that adjoint is defined by

$$(Ax, y) = (x, A^*, y)$$

Can we find the adjoint of L?
Formal Adjoint of $L : L^*$

$$(Lu, v)_{L^2} = (u, L^*v)_{L^2}$$

$$(Lu, v)_L^2 = \int_\Omega -(a^{ij}u_{u_{x_i}})_{x_j}v + b^iu_{x_i}v + cuv\,\mathrm{d}x$$

$$= \int_\Omega a^{ij}u_{x_i}u_{x_j} - \mathrm{div}(v\vec{b})u + cuv\,\mathrm{d}x$$

$$= \int_\Omega -(a^{ij}v_{x_i})_{v_{x_j}}u - b^iv_{x_i}cx\,\mathrm{d}x = \int_\Omega -b^iv_{x_i} - b^i_{x_i} - b^i_{x_i}v + cv\,\mathrm{d}x$$

Define that to be L^*v
Lemma: $R(I + K) = N(I + K^*)$

Proof. $w \in N(I + K^*)o_p$
$\iff (x, (I + K^*)w = 0)\ \forall x \in H$
$\iff ((I + K)x, w) = 0\ \forall x \in H$
$\iff w \in R(I + K)\perp$. □

Theorem 18. For L as defined, for $\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega \in C^1$, then either:

i) $\exists$ weak $H_0^1(\Omega)$ solution to $Lu = 0, u = 0$ on $\partial\Omega$.

ii) $\forall f \in L^2(\Omega)\exists!$ weak $H_0^1(\Omega)$ solution to $Lu = f, u = 0$ on $\partial\Omega$

Furthermore if i holds thereen $Lu = f, u = 0$ on $\partial\Omega$ has a weak solution $\iff$ $(f, v)_{L^2} = 0\forall v$ such that $L^*v = 0, v = 0$ in $\partial\Omega$.

Example: $-u'' - u = \sin x$ and $u(0) = u(\pi) = 0$.
Here $Lu = -u'' - u$. We found $\nexists$ solution.
Note that L has non-trivial nullspace. $\sin x$ is in the nullspace.

$$\int \sin x \cdot \sin x \neq 0$$

Theorem not applicable.

Proof. Recall $\exists\gamma > 0$ such that

$$B_\gamma[u, v] = B[u, v] + \gamma\int_\Omega uv$$

where:

$$B(u, v) := \int_\Omega (a^{ij}u_{x_i}v_{x_j} + b^iu_{x_i}v + cuv)\,\mathrm{d}x$$

We can apply Lax-Milgrim to obtain a unique weak solution to $L_\gamma u = f$ where $L_\gamma u = Lu + \gamma u$. ch that: So $B_\gamma[u, v] = (f, v)\forall v \in H_0^1(\Omega)$
We say $L_\gamma^{-1} = u$ if this holds.

We seek a function $u \in H_0^1(\Omega)$ such that:

$$B_\gamma[u,v] = (f,v)_{L^2} + \gamma(u,v)_{L^2}$$

We want u such that:

$$u = L_\gamma^{-1}(f+\gamma u) = \underbrace{L_\gamma^{-1}(f)}_{=h} + \gamma L_\gamma^{-1}(u)$$

Note: $L_\gamma^{-1} : L^2 \to H_0^1$ or $L_\gamma^{-1} : L^2 \to L^2$
Let $h := L_\gamma^{-1}(f)$.
So we're trying to solve $u - \gamma L_\gamma^{-1}(u) = h$.
Define $K := \gamma L_\gamma^{-1}$
So we want to solve $(I-K)u = h$
$K : L^2 \to L^2$.
We want to use Lax Milgrim. Why must K be bounded?
Let $g \in L^2(\Omega)$, $\frac{1}{\gamma}K(g) = L_\gamma^{-1}(g) =: u$
$\|K(g)\|_{H_0^1} = \|\gamma u\|H_0^1$. $B_\gamma[u,v] = (g,v)\,\forall v$.
Pick $v = u$ then,

$$B_\gamma[u,u] \geq \beta\|u\|_{H_0^1}^2$$

$$\beta\|u\|^2 \leq B_\gamma[u,u] = (g,u) \leq \|g\|_{L^2}\|u\|_{L^2} \leq C_p\|g\|_{L^2}\|u\|_{H_0^1}$$

$$\gamma\beta\|u\|_{H_0^1} \leq \gamma C_p\|g\|_{L^2}$$

$$\|K(g)\|_{L^2} \leq C\|K(g)\|_{H_0^1} \leq \frac{\gamma C_p}{p}\|g\|_{L^2}$$

<u>Claim</u>: $K : L^2 \to L^2$ is compact.
Let $\{g_k\} \subset L^2$ $\|g_k\| \leq C$.

$$\|K(g_k)\|_{H_0^1} \leq \tilde{C}\|g_k\|_{L^2} \leq \text{const}$$

By Rellich-Kondrachov,

$$K(g_{k_j}) \xrightarrow{L^2} W$$

Now, apply Fredholme alternative. Then, $\forall h \in L^2 \exists!$ solution u to $(I-K)u = h$ or else the nullspace of $(I-K^*)$ is non-trivial.

$$(I-K)u = h \iff u \text{ weakly solution to } Lu = f$$

Then apply the Lemma.

□

Friday, 10/4/2024

Recall: Given a bounded linear operator $T : H \to H$, the <u>resolvent set</u>:

$$\rho(T) := \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijective}\}$$

Then the <u>spectrum</u> of T is:

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

$\lambda \in \sigma(T)$ is an <u>eigenvalue</u> if $\exists x \in H$ such that $(T - \lambda I)x = 0$.
$\{\lambda \in \sigma(T) : \lambda$ is not an eigenvalue$\}$ is called the <u>continuous spectrum</u>.
Now, what if $T = K$ compact, linear operator?

We have seen that $I-K$ has a nontrivial nullspace or else it is invertible, and $(I-K)^{-1}$ is bounded.
Further, we have seen that for $\lambda \neq 0$ either $I - \frac{1}{\lambda}K$ has a nontrivial nullspace or it is invertible.
Then, $(I - \frac{1}{\lambda}K)^{-1}$ is bounded.
Nontrivial nullspace implies λ is an eigenvalue of K.
Thus, K compat $\implies$ no continuous spectrum [except perhaps 0].

Theorem 19. A compact operator $K : H \to H$ possesses at most a countable set of eigenvalues having no limit pointsexcept possibly 0.
Furthermore, Each eigenvalue has a finite multiplicity [$\dim N(K - \lambda I)$ is finite].

Proof. Suppose not. Then we have an accumulation point $\exists\{\lambda_n\}$ of eigenvalues such that $\lambda_n \to \lambda \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \{0\}$ and a sequence of linearly independent eigenvectors x_n.
Let $M_n = \operatorname{span}\{x_1, \cdots, x_n\}$.
M_n is a closed subspace.
Projection Lemma $\implies \exists\{y_n\} \subset M_n \setminus M_{n-1}$ so that $\|y_n\| = 1, \operatorname{dist}(y_n, M_{n-1}) \geq \frac{1}{2}$.
Let $S_\lambda \coloneqq \lambda I - K$.
For $n > m$: we have:

$$\lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m = y_n - \lambda_n^{-1} S_{\lambda_n} y_n - y_m + \lambda_m^{-1} S_{\lambda_m} y_m = y_n + z$$

Claim: $z \in M_{n-1}$. To prove this, note that z is sum of elements of M_{n-1}.
$y_m \in M_m \subseteq M_{n-1}$.
Write $y_n = \sum_{j=1}^n c_j x_j$
$S_{\lambda_n} y_n = (\lambda_n I - K)\left(\sum_{j=1}^n c_j x_j\right) = \sum_{j=1}^n (\lambda_n c_j x_j - c_j \lambda_j x_j) = \sum_{j=1}^{n-1} (\lambda_n c_j x_j - c_j \lambda_j x_j)$
Thus $S_{\lambda_n} y_n \in M_{n-1}$
Also $S_{\lambda_m} y_m \in M_{m-1} \subseteq M_{n-1}$.
Now, $y_n = \lambda_n^{-1} K y_n$.
Therefore, $\|\lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m\| \geq \frac{1}{2}$.
$\lambda_n^{-1} \|K y_n - \frac{\lambda_n}{\lambda_m} K y_m\| \geq \frac{1}{2}$.
If λ_m approaches finite value, by taking m, n large enough we get $\|K y_n - K y_m\| \geq \frac{\lambda}{4}$.
Contradiction.
If we have an infinite limit then LHS approaches 0 which is also not possible.

∎

Theorem 20. i) $\exists$ an at most countable set $\Sigma \subset \mathbb{R}$ such that $Lu = \lambda u + f$ in Ω and $u = 0$ on $\partial\Omega$ has a solutio $\forall f \in L^2$ provided $\lambda \neq \Sigma$.

ii) If Σ is infinite then writing $\Sigma = \{\lambda_n\}, \lambda_1 \leq \lambda_2 \leq \cdots$ then $\lambda_n \to \infty$.

iii) If $\lambda \in \Sigma$ then $Lu = \lambda u + f$ for $f \in L^2$ with $u = 0$ on $\partial\Omega$ is solvable $\iff$ $(f, v)_{L^2} = 0 \forall v \in N(L^* - \lambda I)$.

Recall: u is a weak solution to $Lu = \lambda u$ in $\Omega, u = 0$ on $\partial\Omega \iff B[u, v] = \lambda(u, v) \forall v \in H_0^1(\Omega)$.
Where $B[u, v] = \int_\Omega a_{ij}(x) u_{x_i} v_{x_j} + b_i(x) u_{x_i} v + c(x) uv \, \mathrm{d}x$.
Now, $B[u, v] = \lambda(u, v) \iff B_\gamma[u, v] \coloneqq B[u, v] + \gamma \int uv = (\gamma + \lambda)(u, v)$.
Where γ is sufficiently large to make $B_\gamma[u, v] \geq \beta \|u\|_{H_0^1}^2$
So u solves $(P) \iff$

$$u = L_\gamma^{-1}\left((\lambda + \gamma)u\right)$$

$$\iff u = \gamma L_\gamma^{-1}\left(\frac{\lambda + \gamma}{\gamma} u\right) = \frac{\lambda + \gamma}{\gamma} K u \text{ where } K = \gamma L_\gamma^{-1}$$

Thus, u solves $P \iff \left(K - \frac{\gamma}{\gamma + \lambda} I\right) u = 0$.
Then the only posssible limit point of eigenvalues of K is $0 \iff$ only possible limit point of eigenvalues of $L = \infty$.

Monday, 10/7/2024

Weak Convergence

Given a banach space X let X^* be the space of bounded linear functionals.
Given $x^* \in X^*$:

$$\|x^*\|_{X^*} := \sup_{\|x\|_X \leq 1} \langle x^*, x\rangle$$

Definition 18. We say $\{x_n\} \subset X$ converges weakly to $x \in X$ if:

$$\forall x^* \in X^*,\ \langle x^*, x_n\rangle \to \langle x^*, x\rangle$$

Notation: $x_n \overset{X}{\rightharpoonup} x$

Example: for $1 < p < \infty$ If $X = L^p(\Omega)$ then when $\frac{1}{p} + \frac{1}{q} = 1$ we have $X^* \cong L^q(x)$.
$u_n \overset{L^p}{\rightharpoonup} u$ means:

$$\forall v \in L^q,\ \int_\Omega u_n v \,\mathrm{d}x \to \int_\Omega uv \,\mathrm{d}x$$

Facts:

- $x_n \to x \implies x_n \rightharpoonup x$
- $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\| \implies x_n \to x$
- $x_n \rightharpoonup x \implies \|x_n\| \leq M$ for some M.

Definition 19. A reflexive Banach space X is one such that:

$$(X^*)^* = X$$

Example: $L^p(\Omega)$ for $1 < p < \infty$

If X is reflexive and $\|x_n\| \leq M$ then $\exists \{x_{n_j}\}, x \in X$ such that $x_{n_j} \rightharpoonup x$.
If $x_n \rightharpoonup x$ then $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$.

Proof. If $x^* \in X^*, \|x^*\| \leq 1$ then,

$$\liminf_{n\to\infty} \|x_n\| \geq \liminf_{n\to\infty} \langle x^*, x_n\rangle = \langle x^*, x\rangle$$

We see our aswer by taking sup over all $\|x^*\| \leq 1$. $\square$

Basic Question:
Suppose $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ in Ω where L is an elliptic operator and $f \in L^2$ [eg $-\Delta u = f$]. Can one argue that u is better than $H^1(\Omega)$?
A (formal) calculation suggesting that this isn't a ridiculous question:
Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is smooth and compactly supported and solving $-\Delta u = f$ for $f \in L^2$.

$$\begin{aligned}
\infty > \int_{\mathbb{R}^\ltimes} f^2 \,\mathrm{d}x &= \int_{\mathbb{R}^n} (\Delta u)^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} \nabla \cdot (\nabla u) \Delta u \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^n} \nabla u \cdot \nabla(\Delta u) \,\mathrm{d}x = -\int_{\mathbb{R}^n} u_{x_j} \frac{\partial}{\partial x_j}(u_{x_i x_i}) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} u_{x_i x_j} u_{x_i x_j} \,\mathrm{d}x = \int_{\mathbb{R}^n} |D^2 u|^2 \,\mathrm{d}x
\end{aligned}$$

So we have control over all second derivatives of u.

Difference Quotients

Definition 20. Given $u : \underset{\subset \mathbb{R}^n}{\Omega} \to \mathbb{R}$ given $h \in \mathbb{R}$ given $k \in 1, \cdots, n$, define:

$$D_k^h := \frac{u(x + he_k) - u(x)}{h}$$

We want to bound the difference quotient of the derivative to control the second derivaive.
Lemma 1: For $1 < p < \infty$, let $u \in W^{1,p}(\Omega)$ for $\Omega \subset \mathbb{R}^n$. Then $\forall \Omega' \subset\subset \Omega$ and for h such that $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$ we have:

$$\|D_k^h u\|_{L^p(\Omega')} \leq \|u_{x_k}\|_{L^p(\Omega)}$$

Proof. First assume $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$.

$$D_k^h u(x) = \frac{1}{h} \int_0^h u_{x_k} u(x_1, \cdots, x_k + s, \cdots, x_n) \, \mathrm{d}s$$

$$|D_k^h(x)|^p \underset{\text{Hölder}}{\leq} \frac{1}{h^p} \left(\int_0^h 1^{\frac{p}{p-1}} \, \mathrm{d}x \right)^{p-1} \left(\int_0^h |u_{x_k}(\cdots, x_k + s, \cdots)|^p \, \mathrm{d}x \right)$$

$$\int_{\Omega'} |D_k^h u(x)|^p \, \mathrm{d}x \leq \frac{1}{h} \int_{\{\operatorname{dist}(x,\partial\Omega) > h\}} \int_0^h |u_{x_k}(\cdots x_k + s \cdots)|^p \, \mathrm{d}s \, \mathrm{d}x$$

$$\underset{\text{Fubini}}{\leq} \frac{1}{h} \int_0^h \int_\Omega |u_{x_k}(x)|^p \, \mathrm{d}x \, \mathrm{d}s = \|u_{x_k}\|^p_{L^p(\Omega)}$$

Now we approximabe by C^1 functions.

□

Lemma 2: Let $u \in L^p(\Omega)$ for $1 < p < \infty$ and suppose for some $k \in \{1, \cdots, n\}$, $\exists M$ such that

$$\|D_k^h u\|_{L^p(\Omega')} \leq M$$

$\forall \Omega' \subset\subset \Omega, \forall h$ such that $\operatorname{dist}(\Omega', \partial\Omega) > h$. Then,

$$\|u_{x_k}\|_{L^p(\Omega)} \leq M$$

Proof. $\exists \{h_j\} \to 0, \exists v \in L^p(\Omega)$ such that:

$$D_k^{h_j} u \overset{L^p(\Omega)}{\rightharpoonup} v$$

[We are using diagonalization. We cannot directly go near $\partial\Omega$ but by making $h_j \to 0$ and taking subsequences, we get the convergence for whole Ω].
Furthermore, by lower semicontinuity of $\|\cdot\|_{L^p}$ under weak convergence: $\|v\|_{L^p(\Omega)} \leq M$.
Must still sho v is the weak k'th derivative of u.
Fix $\phi \in C_0^1(\Omega)$.
We have:

$$\lim_{h_j \to 0} \int_\Omega \phi D_k^{h_j} u \, \mathrm{d}x = \int_\Omega \phi v \, \mathrm{d}x$$

Now we do integration by parts on difference quotient. We also have:

$$\int_\Omega \phi D_k^{h_j} u \, \mathrm{d}x = \int_\Omega \phi(x) \left(\frac{u(x + h_j e_k) - u(x)}{h_j} \right) \mathrm{d}x$$

For first term, let $y = x + h_j e_k$.

$$= \int_{\operatorname{supp}(\phi + h_j e_k)} \frac{\phi(y - h_j e_k) u(y)}{h_j} \, \mathrm{d}y - \int_{\operatorname{supp}(\phi)} \frac{\phi(x) u(x)}{h_j} \, \mathrm{d}x$$

$$= -\int_\Omega \frac{\phi(y-h_je_k)-\phi(y)}{-h_j}u(y)\,\mathrm{d}y = -\int_\Omega D_k^{-h_j}\phi(y)u(y)\,\mathrm{d}y$$

Now let $h_j \to 0$

$$\implies \lim_{h_j\to 0}\int_\Omega \phi D_k^{h_j}u\,\mathrm{d}x = -\int_\Omega \phi_{x_k}u\,\mathrm{d}x$$

Thus v is the weak x_k derivative of u.

□

Wednesday, 10/9/2024

Recall:

$$D_k^h u(x) \coloneqq \frac{u(x+he_k)-u(x)}{h}$$

$$\int D_k^h u_1(x)u_2(x)\,\mathrm{d}x = -\int D_k^{-h}u_2(x)u_1(x)\,\mathrm{d}x$$

Lemma 1: Suppose $1 < p < \infty$. Let $u \in W^{1,p}(\Omega)$. Then, $\forall \Omega' \subset\subset \Omega$ such that $\operatorname{dist}(\Omega', \partial\Omega) > |h|$ we have:

$$\|D_k^h u\|_{L^p(\Omega)} \le \|u_{x_k}\|_{L^p(\Omega)}$$

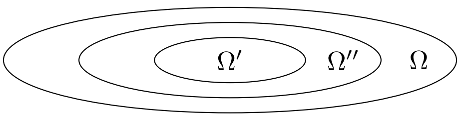
Lemma 2: Let $u \in L^p(\Omega), 1 < p < \infty$. Assume $\exists M > 0$ such that $D_k^h u \in L^p(\Omega')$ and $\|D_k^h u\|_{L^p(\Omega')} \le M\ \forall k, \forall \Omega' \subset\subset \Omega$ such that $\operatorname{dist}(\Omega', \partial\Omega) \ge |h|$.
Then u_{x_k} exists $\forall k$ and $\|u_{x_k}\|_{L^p(\Omega')} \le M$.
Suppose $Lu \coloneqq -\frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + b_i(x)u_{x_i} + c(x)u$
Assume $a_{ij}(x)\zeta_i\zeta_j \ge \theta|\zeta|^2, a_{ij} = a_{ji}$
Assume $a_{ij} \in C^1(\Omega), b_i, c \in L^\infty(\Omega)$

Theorem 21 (Interior Regularity)**.** Assume $u \in H^1(\Omega)$ weak solution to $Lu = f$ in Ω for $f \in L^2(\Omega)$. Then $u \in H^2_{loc}(\Omega)$ with:

$$\|u\|_{H^2(\Omega')} \le C(\Omega', \Omega, \theta, \cdots)\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

$\forall \Omega' \subset\subset \Omega$.

Proof. Consider $\Omega' \subset\subset \Omega'' \subset\subset \Omega$



Let ζ be a smooth cutoff function:
$\zeta \equiv 1$ in Ω'
$\zeta \equiv 0$ on $\Omega \setminus \Omega''$
$0 \le \zeta \le 1$
We have:

$$\int_\Omega a_{ij}(x)u_{x_i}v_{x_j} + b_i(x)u_{x_i}v + c(x)uv\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x$$

$\forall v \in H_0^1(\Omega)$
Note that,

$$\int_\Omega a_{ij}(x)u_{x_i}v_{x_j} + \boxed{b_iu_{x_i}}v + \boxed{cu}v\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x$$

Choose $v = -D_k^{-h}(\zeta^2 D_k^h u)$

in Ω' where $\zeta \equiv 1$ we have:

$$v = -\left(D_k^{-h} \left(\frac{u(x+he_k) - u(x)}{h} \right) \right)$$

$$= -\left(\frac{u(x - he_k + he_k) - u(x - he_k) - u(x + he_k) + u(x)}{-h^2} \right)$$

$$= -\left(\frac{u(x+he_k) + u(x-he_k) - 2u(x)}{h^2} \right)$$

Using this v, we get:

$$I = \int_\Omega -a_{ij} u_{x_i} D_k^{-h} (\zeta^2 D_k^h u)_{x_i} - b_i u_{x_i} D_k^{-h}(\zeta^2) - cu D_k^{-h}(\zeta^2 D_k^h u) \, \mathrm{d}x$$

$$= -\int_\Omega f D_k^{-h} (\zeta^2 D_k^h u) \, \mathrm{d}x$$

$$I = \int_\Omega D_k^h (a_{ij} u_{x_i}) \left(\zeta^2 D_k^h u_{x_i} + 2\zeta \varphi_{x_j} D_k^h u \right) \, \mathrm{d}x$$

$$= \int_\Omega \left(a_{ij}(x + he_k) D_k^h (u_{x_i}) + D_k^h (a_{ij}) u_{x_i} \right) \left(\zeta^2 D_k^h u_{x_j} + 2\zeta \varphi_{x_j} D_k^h u \right) \, \mathrm{d}x$$

$$= \int_\Omega \zeta^2 a_{ij}(x+he_k) D_k^h(u_{x_i}) D_k^h(u_{x_j}) + \text{others } \, \mathrm{d}x$$

$$\geq \theta \int_\Omega \zeta^2 |D_k^h (\nabla u)|^2 + \text{ others } \, \mathrm{d}x$$

Write $\tilde{f} \coloneqq f - b_i u_{x_i} - cu \in L^2$.
Weak form becomes:

$$\int_\Omega a_{ij}(x) u_{x_i} v_{x_j} \, \mathrm{d}x = \int_\Omega \tilde{f} v \, \mathrm{d}x$$

We have:

$$\theta \int_\Omega \zeta^2 |D_k^h (\nabla u)|^2 \, \mathrm{d}x \leq \int_\Omega \tilde{f} D_k^{-h} (\zeta^2 D_k^h u) - a_{ij}(x + he_k) 2\zeta \zeta_{x_j} D_k^h (u_{x_i}) D_k^h u$$

$$- D_k^h (a_{ij}) u_{x_i} \zeta^2 D_k^h u_{x_j} - D_k^h (a_{ij}) u_{x_i} 2\zeta \zeta_{x_j} D_k^h u \, \mathrm{d}x$$

Use $ab \leq \varepsilon^2 a^2 + \frac{1}{4\varepsilon^2} b^2$ to estimate terms 2,3,4.

$$\leq C \int_{\Omega''} \left(|D_k^h (\nabla u)| |D_k^h u| + |D_k^h (\nabla u)| |\nabla u| + |\nabla u| |D_k^h u| \right) \zeta \, \mathrm{d}x$$

Note: $C \to \infty$ as $\Omega'' \to \Omega$ since it involves the derivative of the cutoff function.

Monday, 10/14/2024

Recall: weak formulation:

$$\int_\Omega a_{ij} u_{x_i} v_{x_i} \, \mathrm{d}x = \int_\Omega f v - b_i u_{x_i} - cuv \, \mathrm{d}x$$

$$=: \int_\Omega \tilde{f} v \, \mathrm{d}x, \tilde{f} \in L^2(\Omega)$$

Choose: $v = -D_k^{-h} (\zeta^2 D_k^h u)$
Start with LHS.
We arrived at:

$$LHS = \int_\Omega a_{ij}(x+he_k) D_k^h D_k^h (u_{x_j}) \zeta^2 \,\mathrm{d}x$$

$$+ \int_\Omega \bigg[a_{ij}(x+he_k) 2\zeta\zeta_{x_i} D_k^h u D_k^h(u_{x_i}) + D_k^h(a_{ij}) 2\zeta\zeta_{x_j} D_k^h(u) u_{x_i}$$

$$+ D_k^h(a_{ij}) u_{x_i} D_k^h(u_{x_j}) \zeta^2 \bigg] \,\mathrm{d}x$$

$$\implies LHS \geq \theta \int_\Omega |D^h(\nabla u)|^2 \zeta^2 \,\mathrm{d}x$$

$$-C \left\{ \int_\Omega |D_k^u||D_k^h(\nabla u)| + |D_k^h u||\nabla u| + |D_k^h(\nabla u)||\nabla u| \,\mathrm{d}x \right\}$$

Now use $\left(\varepsilon a - \frac{1}{2\varepsilon} b\right)^2 \geq 0 \implies ab \leq \varepsilon^2 a^2 + \frac{1}{4\varepsilon^2} b^2$
Now use our lemmas.

$$LHS \geq \theta \int |D^h(\nabla u)|^2 - \varepsilon^2 |D_k^h(\nabla u)|^2 - C_\varepsilon |\nabla u|^2$$

Pick $\varepsilon^2 = \frac{\theta}{2}$

$$LHS \geq \frac{\theta}{2} \int_\Omega |D_k^h(\nabla u)|^2 \zeta^2 \,\mathrm{d}x - C_\varepsilon \int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

Now we estimate RHS with a little of L^2 norm v + a lot of L^2 norm $\tilde{f}$
By lemma 1:

$$\int v^2 \leq \int |\nabla(\zeta^2 D_k^h u)|^2$$

$$\leq C \int |D_k^h|^2 + |D_k^h(\nabla u)|^2 \,\mathrm{d}x$$

$$v\tilde{f} \,\mathrm{d}x \leq \varepsilon^2 \int v^2 + C_\varepsilon \int \tilde{f}^2$$

Pick $\varepsilon^2 = \frac{\theta}{4}$

$$\implies \frac{\theta}{4} \int_{\Omega'} \left| D_k^(\nabla u) \right|^2 \,\mathrm{d}x \leq C \int_\Omega (f^2 + |\nabla u|^2 + u^2) \,\mathrm{d}x$$

$$\leq C \int_\Omega f^2 + \|u\|_{H^1(\Omega)}^2 \,\mathrm{d}x$$

Apply Lemma 2
$u \in H_{loc}^2(\Omega)$ and $\forall \Omega' \subset\subset \Omega$

$$\int_{\Omega'} |D^2 u|^2 \,\mathrm{d}x \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right)$$

Finally, we need to replace $\|u\|_{H^1}^2$ with $\|u\|_{L^2}^2$ on RHS.
Using a new cut-off function ζ such that $\zeta \equiv 0$ on $\Omega \setminus \Omega''$, we have:

$$\|u\|_{H^2(\Omega')} \leq C \left(\|f\|_{L^2(\Omega'')}^2 + \|u\|_{H^1(\Omega'')}^2 \right)$$

Now go back to $(*)$ weak formulation:
Choose $v = \zeta^2 u$

$$\int_\Omega a_{ij} u_{x_i} u_{x_j} \zeta^2 + a_{ij} u_{x_i} 2\zeta\zeta_{x_i} \,\mathrm{d}x = \int \tilde{f} \zeta^2 u$$

Again, by ellipticity and $ab \leq \varepsilon a^2 + C_\varepsilon b^2$,
Choosing ε small in terms of θ,

$$\int_{\Omega'} |\nabla u|^2 \, \mathrm{d}x \leq C \int_{\Omega''} f^2 + u^2 \, \mathrm{d}x$$

so we're done!

□

Wednesday, 10/16/2024

Higher Interior Regularity

Suppose $Lu \coloneqq -(a_{ij}u_{x_i})_{x_j} + b_i(x)u_{x_i} + c(x)u$
So far: Assume a_{ij} elliptic, $a_{ij} \in C^1$, $b_i, c \in C^\infty$ $f \in L^2(\Omega)$.
Then if $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ then $u \in H^2_{\mathrm{loc}}(\Omega)$ and $\forall \Omega' \subset\subset \Omega \exists C(\Omega')$ such that:

$$\|u\|_{H^2(\Omega')} \leq C(\Omega')(\|f\|_{L^2} + \|u\|_{L^2})$$

What if a_{ij}, b_i, c are <u>nicer</u>, as is f? Then u should be <u>nicer</u>.
Idea: Consider the PDE satisfied (weakly) by u_{x_k} for some $k \in \{1, \cdots, n\}$.

$$\frac{\partial}{\partial x_k} L(u) = \frac{\partial}{\partial x_k} f$$

$$\begin{gathered} -(a_{ij}(x)(u_{x_k})_{x_i})_{x_j} - (a_{ij}(x))_{x_k}(u_{x_k})_{x_i} + (b_i(x))_{x_k} u_{x_i} + b_i(x)(u_{x_k})_{x_i} \\ +(c(x))_{x_k} u + c(x) u_{x_k} = f_{x_k} \end{gathered}$$

This is not exactly allowed. So we express it weakly.
Then u_{x_k} weakly solves a <u>new</u> elliptic PDE.

Theorem 22. Let m = non-neg integer. Assume $a_{ij}, b_i, c \in C^{m+1}(\Omega)$. Assume $f \in H^m(\Omega)$
Ten if $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ we have:

$$u \in H^{m+2}_{\mathrm{loc}}(\Omega)$$

and $\forall \Omega' \subset\subset \Omega$

$$\|u\|_{H^{m+2}(\Omega')} \leq C(\|f\|_{H^m} + \|u\|_{L^2})$$

Proof. By induction. □

Boundary Regularity

Theorem 23. Take $\Omega \subset \mathbb{R}^n$, open, bounded, $\partial\Omega \in C^2$. Take a_{ij} elliptic, $a_{ij} \in C^1, b_i, c \in L^\infty$.
Take $f \in L^2$.
Then if $u \in H^1_0(\Omega)$ is a weak solution to $Lu = f$ in Ω and $u = 0$ on $\partial\Omega$ then,

$$u \in H^2(\Omega)$$

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2} + \|u\|_{L^2})$$

Proof. First we assume the boundary is flat. If not we flatten the boundary
Suppose first that $B(0,1) \cap \Omega = B(0,1) \cap \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$.

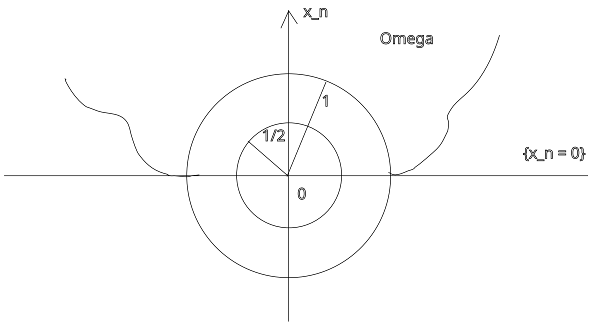


Figure 2:

Let ζ be a cutoff function.
$\zeta \in C^\infty, \zeta \equiv 1$ in $B(0, \frac{1}{2})$
$\zeta \equiv 0$ in $\mathbb{R}^n \setminus B(0,1)$

$$B[u,v] = \int fv \, \forall v \in H_0^1(\Omega)$$

Rewrite:

$$\int_\Omega a_{ij} u_{x_i} v_{x_j} \, \mathrm{d}x = \int_\Omega \widetilde{f} v \, \mathrm{d}x$$

For $|h|$ small take:

$$v = -D_k^{-h}(\zeta^2 D_k^h u)$$

for any $k \in \{1, \cdots, n-1\}$
To be a legal v for this, we need $v \in H_0^1(\Omega)$ in light of the cut-off function and $\mathrm{Tr}\big|_{\partial B(0,1)^+} u = 0$
So we can use this v in $*$.
The rest of the proof is the same for interior regularity. We obtain:

$$\|D_k^h(\nabla u)\|_{L^2(B(0,1)^+)} \leq C\left(\|f\|_{L^2} + \|u\|_{L^2}\right)$$

Since difference quotients are uniformly bounded we must have weak derivatives.
$\implies$ by lemma 2,

$$\|u_{x_i x_j}\| \leq C\left(\|f\|_{L^2} + \|u\|_{L^2}\right)$$

for all i, j except $i = j = n$.
How to control $u_{x_n x_n}$?
By interior regularity, $Lu = f$ at a.e. $x \in \Omega$.
We rewrite the PDE:

$$-a_{nn} u_{x_n x_n} = \underbrace{\sum_{(i,j) \neq (n,n)} (a_{ij}(x) u_{x_i})_{x_j} - b_i(x) u_{x_i} - c(x) u + f}_{:= \tilde{\tilde{f}}}$$

We have $\|\tilde{\tilde{f}}\| < \mathrm{const}(\|f\|_{L^2} + \|u\|_{L^2})$

$$\implies \|u_{x_n x_n}\| \leq \frac{\mathrm{const}}{\theta}\left(\|f\|_{L^2} + \|u\|_{L^2}\right)$$

Friday, 10/18/2024

We have proved the theorem for flat portion of $\partial\Omega$. For proving the general case, we flatten the boundary.
Locally write $\partial\Omega$ as a graph:

$$x_n = f(x_1, \cdots, x_{n-1})\, f \in C^2$$

We change variables.

$$y_j = x_j\, j = 1, \cdots, n-1$$

$$y_n = x_n - f(x_1, \cdots, x_{n-1})$$

$$y = \Phi(x)$$

[insert picture figure]
Then,

$$D\Phi(x) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -f_{x_1} & -f_{x_2} & \cdots & 1 \end{bmatrix}$$

$\det D\Phi = J\Phi = 1$.
By inverse function theorem, Φ is invertible. Let $\Psi = \Phi^{-1}$. Define:

$$\widetilde{u}(y) \coloneqq u(\Psi(y))$$

$$u(x) = \widetilde{u}(\Phi(x))$$

By chain rule,

$$u_{x_i} = \widetilde{u}_{y_k} \Phi^{(k)}_{x_i}$$

Weak form:

$$\int_{\Phi(B(x_0,R)\cap\Omega)} \left(\underbrace{a_{ij}(\Psi(y))\widetilde{u}_{u_k}\Phi^{(k)}_{x_k}\widetilde{v}_{y_l}\Phi^{(l)}_{x_j}}_{\widetilde{a}_{kl}\widetilde{u}_{y_k}\widetilde{v}_{y_l}} + b_i(\Psi(y))\widetilde{u}_{x_k}\Phi^{(k)}_{x_i} + c(\Psi(y))\widetilde{u} \right) \cdot 1 \,\mathrm{d}y$$

$$= \int_{\Phi(B(x_0,R)\cap\Omega)} f(\psi(y))\widetilde{v}(y)\,\mathrm{d}y$$

<u>Claim</u>: $\widetilde{u}$ solves an elliptic PDE weakly on a <u>flat</u> domain.
Define $\widetilde{a}_{kl}(y) \coloneqq a_{ij}\Phi^{(k)}_{x_i}\Phi^{(l)}_{x_j}$.
We're done one we check that:

$$\widetilde{a}_{kl} = \widetilde{a}_{lk}, \widetilde{a_{kl}}\eta_k\eta_l \geq \widetilde{\theta}|\eta|^2\, \forall \eta \in \mathbb{R}^n$$

Firstly,

$$\widetilde{a}_{lk} = a_{ij}\Phi^{(l)}_{x_i}\Phi^{(k)}_{x_j} = a_{ji}\Phi^{(l)}_{x_i}\Phi^{(k)}_{x_j} = \widetilde{a}_{kl}$$

Then let $\eta \in \mathbb{R}^n$.

$$\widetilde{a}_{kl}\eta_k\eta_l = a_{ij}\Phi^{(k)}_{x_i}\Phi^{(l)}_{x_j}\eta_k\eta_l = a_{ij}\underbrace{\Phi^{(k)}_{x_i}\eta_k}_{=\xi_i}\underbrace{\Phi^{(l)}_{x_j}\eta_l}_{=\xi_j} \geq \theta|\xi|^2$$

Let $\xi_i = \Phi^{(k)}_{x_i}\eta_k$ and $\xi_j = \Phi^{(l)}_{x_j}\eta_l$.
Then, $\xi = D\Phi\eta \implies (D\Phi)^{-1}\xi = \eta \implies D\Psi\xi = \eta$.

$$|\eta| \leq |D\Psi||\xi| \leq C_{d\Omega}|\xi|$$

Therefore,

$$\widetilde{a}_{kl}\eta_k\eta_l \geq \theta|\xi|^2 \geq \frac{\theta}{C_{\partial\Omega}}|\eta|^2$$

So, this reduces to the case of flat boundary, which we have already proven.

□

Maximum Principles

Consider L in non-divergence form:

$$Lu \coloneqq -a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} + c(x)u$$

Theorem 24 (Weak Maximum Principle)**.** Assume u is a 'nice classical solution', meaning $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for some bounded open $\Omega \subset \mathbb{R}^n$.
Further assume $c(x) \equiv 0$ in the definition of L.

i) If $Lu \leq 0$ in Ω then,
$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

ii) If $Lu \geq 0$ in Ω then,
$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

Remarks:

1) If $Lu \leq 0$ we call u a subsolution. If $Lu \geq 0$ we call u a supersolution.

2) If $Lu = 0$ then both maximum and minimum are achieved on the boundary.

3) A weak maximum principle does not preclude the max also being achieved inside Ω.

4) If $c \not\equiv 0$, the maximum principle may fail.
   Example: If $\Omega = (0, \pi)$ and $Lu = -u'' - u = 0$ then $a_{11} = 1, b_1 = 0, c(x) = -1$. For Dirichlet boundary condition $u(0) = u(\pi) = 0$. Our answer can be $\sin x$. Then the maximum happens at $\frac{\pi}{2}$ not in boundary for $A > 0$.

Monday, 10/21/2024

Recap:

$$Lu \coloneqq -a_{ij}(x)u_{x_ix_j} + b_i()u_{x_i} + c(x)u_i$$

a_{ij} uniformly elliptic $a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2, \theta > 0, a_{ij} = a_{ji}.a_{ij}, b_i, c$ are continuous on $\overline{\Omega}, \Omega \subset \mathbb{R}^n$ is open bounded.
Then we have the weak maximum principle as shown above.

Proof. Case 1: Suppose $Lu < 0$ [strictly less than 0]. We proceed by contradiction to show a strong maximum principle.
Suppose $\exists x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u(x)$.
Consider $Lu(x_0)$. Note that $u_{x_i}\big|_{x_0} = 0$ since $\nabla u(x_0) = 0$. Thus,

$$Lu(x_0) = -a_{ij}(x_0)u_{x_ix_j}(x_0)$$

Linear Algebra Fact: If a matrix A is symmetric positive definite then A can be diagonalizable by an orthogonal matrix $\mathcal{O}$ so that $\mathcal{O}\mathcal{O}^T = I$. Then,

$$\mathcal{O}A\mathcal{O}^T = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\lambda_1, \cdots, \lambda_n > 0$$

Assume $u \in C^2$ and has a min at x_0.
Change variables: $y = x_0 + \mathcal{O}(x - x_0)$. Then,

$$u_{x_i} = u_{y_k}\mathcal{O}_{ki}$$

$$u_{x_i x_j} = u_{y_k y_l}\mathcal{O}_{ki}\mathcal{O}_{lj}$$

$$a_{ij}(x_0)u_{x_i x_j}(x_0) = \sum_{i,j}\left[\sum_{k,l} a_{ij}(x_0)u_{y_k y_l}\mathcal{O}_{ki}\mathcal{O}_{lj}\right]$$

$$= \sum_{k,l} u_{y_k y_l}\left[\sum_{i,j} a_{ij}(x_0)\mathcal{O}_{ki}\mathcal{O}_{lj}\right]$$

$$= \sum_{k,l} u_{y_k y_l}(\mathcal{O}A)\mathcal{O}^T_{jl}u_{y_k y_l}(x_0) = \lambda_1 u_{y_1 y_1} + \cdots + \lambda_n u_{y_n y_n}$$

At a max, $u_{y_j y_j} \leq 0$ for all j. Then $Lu(x_0) \geq 0$ which gives us the contradiction.
Case 2: Suppose $Lu \leq 0$. We pertrub L to get back to the first case. For example:

$$u^\epsilon(x) := u(x) + \epsilon e^{\lambda x_1}$$

$$Lu^\epsilon = \underbrace{Lu}_{\leq 0} + L(\epsilon e^{\lambda x_1}) \leq -\epsilon\lambda^2 a_{11}(x)e^{\lambda x_1} + \epsilon b_1(x)\lambda e^{\lambda x_1}$$

$$= \epsilon\lambda e^{\lambda x_1}(-\lambda a_{11}(x) + b_1(x)) \leq \epsilon e^{\lambda x_1}(-\theta\lambda + \|b\|_{L^\infty})$$

Pick λ big enough so that $Lu^\epsilon < 0$. By case 1,

$$\max_{\overline{\Omega}} u \leq \max_{\overline{\Omega}} u^\epsilon(x) = \max_{\partial\Omega} u^\epsilon(x) \leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} e^{\lambda x_1} \leq \max_{\partial\Omega} u + \epsilon e^{\lambda R}$$

where $\Omega \subset B(0, R)$. Let $\epsilon \to 0$ to finish the proof.

$\square$

Now we try to make sense of the case $c(x) \geq 0$.

Theorem 25 (Weak max princ. for $c(x) \geq 0$)**.** Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Assume $c(x) \geq 0 \forall x \in \Omega$.

i) If $Lu \leq 0$ in Ω then $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+$

ii) If $Lu \geq 0$ in Ω then $\min_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-$

Where $u^+(x) := \max(u(x), 0), u^-(x) = -\min(u(x), 0)$.

Note: If $Lu = 0$ had a solution, then $\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|$
Example: Let $Lu = -u'' + (4x^2 + 1)u$.
$b_i \equiv 0, c(x) = 4x^2 + 1 > 0$
$\Omega = (-1, 1)$.
Consider $u(x) = e^{-x^2} - 4$
$u' = -2xe^{-x^2}$
$u'' = -2e^{-x^2} + 4x^2 e^{-x^2}$
$Lu = (2 - 4x^2)e^{-x^2} + (4x^2 + 1)(e^{-x^2} - 4) = 3e^{-x^2} - 16x^2 - 4 < 0$ on $(-1, 1)$
Max of u comes in the origin, which is -3. But it is not at the boundary! We need to be careful about the sign and positive part and negative parts.

Proof. Let $\Omega' = \{x \in \Omega : u(x) > 0\}$.
$\Omega' = \varnothing \implies u(x) \le 0$ in $\overline{\Omega}$ so we're done.
If $\Omega' \neq \varnothing$ then we have:
[picture figure]
Let $Ku := Lu - c(x)u$, and K doesn't have the c term. So we can apply theorem 1 to Ω'.

$$Ku \le -c(x)u \le 0 \, \text{in } \Omega'$$

$$\max_{\overline{\Omega}} u \le \max_{\overline{\Omega'}} u \le \max_{\partial\Omega'} u = \max_{\partial\Omega} u^+$$

In $\partial\Omega$ for the negative part $u^+ \equiv 0$ so we can ignore that boundary, we're done!

□

Wednesday, 10/23/2024

Lemma 1 (Hopf Lemma)**.** Assume $u \in C^2(\Omega) \cap C^1(\Omega)$ and assume $c(x) \equiv 0$ Suppose $Lu \le 0$ in Ω and $\exists x_0 \in \partial\Omega$ such that:
$u(x_0) > u(x) \forall x \in \Omega$and
Ω satisfies an interior ball condition at x_0, namely $\exists y_0 \in \Omega, \exists r > 0$ such that $B(y_0, r) \subset \Omega$ with $x_0 \in \partial B(y_0, r)$.
Then $\frac{\partial u}{\partial \nu}(x_0) > 0$ where $\nu = \frac{x_0 - y_0}{|x_0 - y_0|}$ [outer normal].
If $c(x) \ge 0$ then the same conclusion holds provided $u(x_0) \ge 0$.
[pictures / fig bad example]
A <u>sufficient</u> condition for this to hold: if the boundary is given by a C^2 function [so the curvature never gets too extreme] it is enough for the boundary to be C_2.

<u>Note</u>: $\frac{\partial u}{\partial \nu}(x_0) \ge 0$ is immediate since $u(x_0) > u(x) \forall x \in \Omega$.
So the significance is the <u>strict inequality</u>.

Proof. Define $v(x) = e^{-\lambda|x|^2} - e^{-\lambda r^2}$, $\lambda > 0$ to be specified later.

$$v_{x_i} = -2\lambda x_i e^{-\lambda|x|^2}$$

$$v_{x_i x_j} = (4\lambda^2 x_i x_j - 2\lambda\delta_{ij})e^{-\lambda|x|^2}$$

$$Lv = -a_{ij}(x)(4\lambda^2 x_i x_j - 2\lambda\delta_{ij})e^{-\lambda|x|^2} - 2\lambda b_i(x) x_i e^{-\lambda|x|^2} + c(x)e^{-\lambda|x|^2} - c(x)e^{-\lambda r^2}$$

$$\implies Lv \le \left(-4\lambda^2\theta|x|^2 + 2\lambda \operatorname{Tr} A + 2\lambda|b|_{L^\infty}|x| + |c|_{L^\infty}\right) e^{-\lambda|x^2|} - \cancel{c(x)e^{-\lambda r^2}}$$

This is < 0 for some choice of $\lambda = \lambda(\theta, \Omega, \operatorname{Tr} A, |b|_{L^\infty}, |c|_{L^\infty})$
WLOG $y_0 = 0$. Consider the annulus:

$$\left\{x : \frac{r}{2} < |x| < r\right\} = \mathcal{A}$$

[insert picture figure]
$\exists \varepsilon > 0$ such that:

$$u(x_0) \ge u(x) + \varepsilon v(x) \, \forall x \in \partial B\left(0, \frac{r}{2}\right)$$

since $u(x_0) > \max_{\partial B(0, \frac{r}{2})} u$
On $\partial B(0, r)$:

$$u(x_0) \ge u(x) + \varepsilon v(x) = u(x)$$

Note $L(u + \varepsilon v - u(x_0))$:

$$= \underset{\le 0}{Lu} + \varepsilon \underset{<0}{Lv} - c(x)\underset{\le 0}{u(x_0)} < 0$$

And we've shown that $u(x) + \varepsilon v(x) - u(x_0) \le 0$ on $\partial\mathcal{A}$.
By weak maximum principle,

$$u(x) + \varepsilon v(x) - u(x_0) \le 0 \,\text{in}\ \mathcal{A}$$

Also note:

$$u(x_0) + \varepsilon v(x_0) - u(x_0) = 0$$

Therefore,

$$\frac{\partial}{\partial \nu}(u(x) + \varepsilon v(x) - u(x_0)) \ge 0$$

$$\implies \frac{\partial}{\partial \nu} u(x_0) \ge -\varepsilon \frac{\partial}{\partial \nu} v(x_0)$$

Note that v is a radial function so $(-\varepsilon)\frac{\partial}{\partial \nu} v(x_0) = 2\varepsilon\lambda(x_0)e^{-\lambda r^2} > 0$.

□

Theorem 26 (Strong Maximum Principle)**.** Assume $\Omega \subset \mathbb{R}^n$ is open, bounded and connected. Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.
Suppose $c \equiv 0$.

i) If $Lu \le 0$ in Ω and if u attains its maximum over $\overline{\Omega}$ at an interior point, then $u \equiv$ const.

ii) If $Lu \ge 0$ in Ω and if u attains its minimum over $\overline{\Omega}$ at an interior point then $u \equiv$ const.

Proof. Let $M := \max_{\overline{\Omega}} u$. Let $S := \{x \in \Omega : u(x) = M\}$.
If $S = \Omega$ we're done.
If $S = \varnothing$ then we're done.
So suppose, by contradiction, $S \ne \Omega, \varnothing$.
[insert picture]
Choose $y \in \Omega \setminus S$ so that:

$$\operatorname{dist}(y, S) < \operatorname{dist}(y, \partial\Omega)$$

Draw the largest open ball B centered at y that doesn't intersect S.
Necessarily, $\exists x_0 \in \partial B \cap S$.
Thus $\Omega \setminus S$ satisfies an interior ball condition at x_0.
$u(x_0) > u(x) \forall x \in \Omega \setminus S$.
We apply Hopf lemma.
The outer normal derivative $\frac{\partial u}{\partial \nu}(x_0) > 0$.
This cannot be true, since $\frac{\partial u}{\partial \nu}(x_0) = \nabla u(x_0) \cdot \nu = 0$ since $\nabla u(x_0) = 0$ at an internal max. □

Friday, 10/25/2024

Theorem 27 (Strong Maximum Principle with $c(x) \ge 0$)**.** Assume $\Omega \subset \mathbb{R}^n$, bounded open with $\partial\Omega \subset C^2$. Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

i) If $Lu \le 0$ in Ω, u achieves a non-negative max inside Ω then $u \equiv$ const.

ii) If $Lu \ge 0$ in Ω, u achieves a non-positive min inside Ω then $u \equiv$ const.

Proof. Identical to $c \equiv 0$ case. □

Uniqueness

For $Lu = f, c(x) \geq 0, u = 0$ on $\partial\Omega$ we have seen a uniqueness result.

Theorem 28. $\Omega \subset \mathbb{R}^n$ open bounded an $\partial\Omega \in C^2$.Suppose u_1 and u_2 both solve $Lu = f$ in Ω with $c(x) \equiv 0$ and $\nabla u \cdot \nu = g$ on $\partial\Omega$ where $\nu =$ outer unit normal.
Then $u_1 - u_2 \equiv$ const.

Proof. Let $v := u_1 - u_2$.
Then $Lv = Lu_1 - Lu_2 = f - f = 0$ in Ω.

$$\nabla v \cdot \nu = \nabla u_1 \cdot \nu - \nabla u_2 \cdot \nu = g - g = 0 \,\text{on}\, \partial\Omega$$

By the max principle either $v \equiv$ const or v attains its max at a point $x_0 \in \partial\Omega$.
Then, $v(x_0) > v(x) \forall x \in \Omega$.
We can use Hopf Lemma:

$$\nabla v(x_0) \cdot \nu > 0$$

this is a contradiction, so we're done. $\square$

Theorem 29. Assume $Lu \leq f$ in a open connected bounded domain $\Omega \subset \mathbb{R}^n$ and assume $c(x) \geq 0$.
Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then,

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_1 \max_{\overline{\Omega}} f^+$$

where C_1 depends on the coefficients of L and Ω.
If $Lu = f$ then we can say

$$\max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |u| + C_1 \max_{\overline{\Omega}} |f|$$

Proof. We use a Barrier construction.
Without loss of generality let's assume

$$\Omega \subset \{x \in \mathbb{R}^n : 0 < x_1 < d\}$$

for some d.
Let $Ku := Lu - cu = -a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i}$.
For $\lambda > 0$ to be chosen let's compute

$$K(e^{\lambda x_1}) = (-a_{11}(x)\lambda^2 + b_1\lambda)e^{\lambda x_1}$$

$$\leq (-\theta\lambda^2 + |b|_{L^\infty}\lambda)e^{\lambda x_1}$$

$$= -\theta\left(\lambda^2 - \frac{\lambda|b|_{L^\infty}}{\theta}\right)e^{\lambda x_1}$$

$$\leq -\theta \,\text{for}\, \lambda \text{ large enough}$$

Goal: Pick a v such that $L(u - v) \leq 0, u - v \leq 0$ on $\partial\Omega$.
Pick $v(x) = \max_{\partial\Omega} u^+ + \left(\frac{e^{\lambda d} - e^{\lambda x_1}}{\theta}\right) \max_{\overline{\Omega}} f^+$
$Lv = Kv + cv \geq Kv$

$$= \frac{\max_{\overline{\Omega}} f^+}{\theta} K(e^{\lambda x_1}) \geq \max f^+$$

Note: $cv \geq 0$ since $c \geq 0, v \geq 0$.
Then, $L(u - v) \leq f - \max f^+ \leq 0$.
On $\partial\Omega$,

$$u - v = u - \max_{\partial\Omega} u^+ - \text{positive} \leq 0$$

By maximum principle,

$$\max_{\overline{\Omega}}(u-v) \leq \max_{\partial\Omega}(u-v)^+ \leq 0$$

Thus, $u - v \leq 0 \implies u \leq v \leq \max_{\partial\Omega} u^+ + \frac{e^{\lambda d}}{\theta} \max_{\overline{\Omega}} f^+$.
Choosing $C_1 = \frac{e^{\lambda d}}{\theta}$ solves our problem. $\square$

Recall that we had

$$\|u\|_{H^2(\Omega')} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

$\forall \Omega' \subset\subset' \Omega, C = C(\Omega', \Omega)$.
If, for example $Lu = f$ in Ω and $u = h$ on $\partial\Omega$ then,

$$|u| \leq \text{const}(f, h)$$

Wednesday, 10/28/2024

[to be entered]

Eigenvalues and Eigenfunctions of Symmetric Elliptic Operators

$Lu = -(a_{ij}(x) u_{x_i})_{x_j} + c(x)u$
$a_{ij} = a_{ji}$
$a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$, $\theta > 0$.
If we included $b_i(x)u_{x_i}$ then $(Lu, v)_{L^2} = \int Lu \cdot v \neq \int uLv$.
From our compact operator approach, we've seen that the set of eigenvalues $\{\lambda_j\}$ is either finite or $\lambda_j \to \infty$.
We also saw: for $a_{ij} = \delta_{ij}$, $c \neq 0$, if there exists a minimizer of the Rayleigh quotient

$$\lambda_1 := \inf_{u \in H_0^1(\Omega), u \not\equiv 0} \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_\Omega u^2 \, \mathrm{d}x} = \mathcal{R}(u)$$

say, u_1, and if u_1 is smooth, then u_1 is the 1st eigenfunction and λ_1 is the smallest eignevalue.
$-\Delta u_1 = \lambda_1 u_1$ in Ω and $u_1 = 0$ on $\partial\Omega$.
Recall the computed:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{R}(u_1 + tv) = 0$$

for any fixed $v \in H_0^1$.

Theorem 30. Given $\Omega \subset \mathbb{R}^n$, bounded, open there exists a function u_1 minimizing $\mathcal{R}(u)$ over all $u \in H_0^1(\Omega), u \not\equiv 0$ Furthermore, u_1 is smooth and is the 1st eigenfunction.

Proposition 2. If $u_j \stackrel{H^1}{\rightharpoonup} u$ then $\liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 \, \mathrm{d}x \geq \int_\Omega |\nabla u|^2 \, \mathrm{d}x$.

Recall $u_j \stackrel{H^1}{\rightharpoonup} u$ means $\forall v \in H^1, (u_j, v)_{H^1} \to (u, v)_{H^1}$.

Proof. Note: Weakly convergent sequences are bounded. Therefore,
$\|u_j\|_{H^1} < C$.
$\implies u_{j_k} \stackrel{L^2}{\to} u$
$\implies \int u_{j_k} v \to \int uv$
$\forall v \in H_0^1, \|v\|_{H_0^1} = \left(\int |\nabla v|^2\right)^{1/2}$
$= \sup_{\|w\|_{H_0^1} \leq 1} \int \nabla v \cdot \nabla w$.
So we're assuming $u_j \rightharpoonup u$.

Fix any w so that $\|w\|_{H_0^1} \leq 1$. We have:

$$\int_\Omega |\nabla u_j|^2 \, \mathrm{d}x \geq \left(\int_\Omega \nabla u_j \cdot \nabla w \, \mathrm{d}x \right)^2$$

Taking $\liminf_{j\to\infty}$,

$$\liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 \, \mathrm{d}x \geq \liminf_{j\to\infty} \left(\int_\Omega \nabla u_j \cdot \nabla w \, \mathrm{d}x \right)^2 = \left(\int_\Omega \nabla u \cdot \nabla w \, \mathrm{d}x \right)^2$$

Now take sup over all such w.

□

Proof of Theorem. We use the direct method.

$$\inf_{u \in H_0^1(\Omega), u \not\equiv 0} \mathcal{R}(u) =: \lambda_1$$

Note, for all $c \neq 0$ constant we have $\mathcal{R}(cu) = \mathcal{R}(u)$.
Thus, WLOG we can minimize $\mathcal{R}$ over the admissible set:
$\mathcal{A} = \left\{ u \in H_0^1(\Omega) : u \not\equiv 0, \int_\Omega u^2 \, \mathrm{d}x = 1 \right\}$.
Let $\{u_j\} \subset \mathcal{A}$ such that $\mathcal{R}(u_j) = \int_\Omega |\nabla u_j|^2 \, \mathrm{d}x \to \lambda_1$.
$\{u_j\}$ is a minimizing sequence.
We know, $\int_\Omega |\nabla u_j|^2 \, \mathrm{d}x < \lambda_1 + 1$.
Thus, $\|u_j\|_{H_0^1} \leq \text{const}$
Thus, there exists subsequence $\{u_{j_k}\}$ such that $u_{j_k} \overset{H^1}{\rightharpoonup} u, u_{j_k} \overset{L^2}{\rightharpoonup} u$ for some $u \in H_0^1(\Omega)$ by Rellich-Kondrachov.
Thus, $\int u^2 = \lim_{k\to\infty} \int u_{j_k}^2 = 1$.
Thus, $u \in \mathcal{A}$.
Thus, by the proposition,

$$\lambda_1 = \liminf_{k\to\infty} \int_\Omega |\nabla u_{j_k}|^2 \, \mathrm{d}x \geq \int_\Omega |\nabla u|^2 \, \mathrm{d}x$$

$$\implies \mathcal{R}(u) = \lambda_1$$

□

If $Lu = -\Delta u + c(x)u$ then,
$\mathcal{R}(u) = \int |\nabla u|^2 + c(x)u^2$.

$$\liminf_{j\to\infty} \mathcal{R}(u_j) \geq \mathcal{R}(u)$$

Wednesday, 10/30/2024

Last time:
We found first eigenfunction u_1 solving

$$\inf_{\substack{u \in H_0^1(\Omega) \\ u \not\equiv 0}} \mathcal{R}(u) = \lambda_1$$

Where $\mathcal{R}(u) = \frac{\int |\nabla u|^2}{\int u^2}$ is the Rayleigh Quotient.
$-\Delta u_1 = \lambda_1 u_1$ in Ω
$u_1 = 0$ on $\partial\Omega$
Today: Higher eigenvalues and eigenfunctions.

$$\lambda_2 := \inf_{\substack{u \in H_0^1(\Omega) \\ \int u^2 = 1 \\ \int_\Omega u u_1 = 0}} \mathcal{R}(u)$$

In general, let $\mathscr{A}_m := \left\{ u \in H_0^1(\Omega) : \int_\Omega u^2 = 1, \int_\Omega u u_j = 0 \text{ for } j = 1, \cdots, n-1 \right\}$

Where u_j minimizes $\inf_{u\in\mathscr{A}_j} R(u) =: \lambda_j$.
$\mathscr{A}_j$ are also sobolev spaces. We can use the exact same proof via the Direct Method to produce a minimizer.
Let $\{v_j\}$ be a miniizing sequence, meaning $R(v_j) \to \lambda_2, \{v_j\} \subset \mathscr{A}_2$, $\|v_j\|_{H_0^1} < \lambda_2 + 1$.
Take $v_{j_k} \overset{H^1}{\rightharpoonup} u_2$
Then $v_{j_k} \overset{L_2}{\to} u_2$
By lower semicontinuity, since $\liminf_{k\to\infty}\|v_{j_k}\|_{H_0^1} \geq \|u_2\|_{H_0^1}$
Thus $\mathcal{R}(u_2) = \lambda_2$
u_2 minimizes $\mathcal{R}(u)$ in $\mathscr{A}_2$
Take first variation:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(u_2 + tv)\bigg|_{t=0} \implies \int_\Omega \nabla u_2 \cdot \nabla v \,\mathrm{d}x = \lambda_2 \int_\Omega u_2 v \,\mathrm{d}x$$

$\forall v \in \mathscr{A}_2$.
Let $\widetilde{v}$ be any element of $H_0^1(\Omega)$ not necessarily $\mathscr{A}_2$. Consider:

$$\int_\Omega \nabla u_2 \cdot \nabla \widetilde{v} \,\mathrm{d}x$$

Write $\widetilde{v} = c_1 u_1 + \hat{v}$ where $c_1 = \int \widetilde{v} u_1 \,\mathrm{d}x, \hat{v} \in \mathscr{A}_2$.

$$\begin{aligned}
\int_\Omega \nabla u_2 \cdot \nabla \widetilde{v} \,\mathrm{d}x &= c_1 \int_\Omega \nabla u_2 \cdot \nabla u_1 \,\mathrm{d}x + \int_\Omega \nabla u_2 \cdot \nabla \hat{v} \,\mathrm{d}x \\
&= -c_1 \int_\Omega u_2 \Delta u_1 \,\mathrm{d}x + \int_\Omega \nabla u_2 \cdot \nabla \hat{v} \,\mathrm{d}x \\
&= c_1 \lambda_1 \cancel{\int_\Omega u_2 u_1 \,\mathrm{d}x} + \int_\Omega \nabla u_2 \cdot \nabla \hat{v} \,\mathrm{d}x \\
&= \lambda_2 \int_\Omega u_2 \hat{v} \,\mathrm{d}x \\
&= \lambda_2 \int_\Omega u_2 \hat{v} \,\mathrm{d}x + \cancel{\lambda_2 \int_\Omega u_2 (c_1 u_1) \,\mathrm{d}x} \\
&= \lambda_2 \int_\Omega u_2 \widetilde{v} \,\mathrm{d}x \,\forall \widetilde{v} \in H_0^1
\end{aligned}$$

So the identity holds outside the subspace. Not only for $\mathscr{A}_2$, but for $\mathscr{A}_j$ by induction.
So we can find λ_j for $j = 1, 2, 3, \cdots$.
By compact operator approach we know either finitely many λ_js or else $\lambda_j \to \infty$.
We claim that eigenvalues cannot have infinite multiplicity.
Note: we can have multiplicity > 1. But it must be always finite. Why?

Theorem 31. Multiplicity is finite.

Proof. Suppose false. Then we have $\{u_j\}$ so that $\mathcal{R}(u_j) = \overline{\lambda}$, $\int u_j^2 = 1, \int u_j u_k = 0, \int |\nabla u_j|^2 = \overline{\lambda}$.
By Kondrachov-Rellich,

$$u_{j_l} \overset{L^2}{\to} \overline{u}$$

for some $\overline{u}$.
It cannot happen, since:

$$\int |u_{j_l} - u_{j_{l'}}|^2 = \int u_{j_l}^2 + u_{j_{l'}}^2 - 2u_{j_l} u_{j_{l'}} = 2$$

so we have $2 = 0$. Contradiction.

□

Theorem 32. $\{u_j\}$ form an orthonormal basis for $L^2(\Omega)$.

That is, $\forall v \in L^2$, we have:

$$v - \sum_{j=1}^{m} c_j u_j \xrightarrow{L^2} 0$$

as $m \to \infty$ where $c_j := \int u_j v$
In 1D, $L(u) = -u'', u(0) = u(b) = 0$.
We are looking at $u'' + \lambda u = 0$. So our functions are:
$u(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right) = 0$.
$u(0) = 0 \implies c_1 = 0$.
$u(b) = 0 \implies \sin\left(\sqrt{\lambda} b\right) = 0$
So $\sqrt{\lambda_j} b = \pi j \implies \lambda_j = \frac{\pi^2 j^2}{b^2}$

Friday, 11/1/2024

Proof. First assume $v \in H_0^1(\Omega)$ [this is dense in $L^2(\Omega)$ so proving in here is enough].
Define for $m \in \mathbb{Z}^+$,

$$v_m = \sum_{j=1}^{m} c_j u_j$$

Where $c_j = \int_\Omega v u_j \, \mathrm{d}x$.
Let $w_m := v - v_m$.
We want to show $w_m \to 0$ in L^2 as $m \to \infty$.
For any $k \leq m$:

$$\langle w_m, u_k \rangle_{L^2} = \int_\Omega w_m u_k \, \mathrm{d}x = \underbrace{\int_\Omega v u_k \, \mathrm{d}x}_{=c_k} - \underbrace{\int_\Omega v_m u_k \, \mathrm{d}x}_{=c_k} = 0$$

Thus, w_m is admissible for

$$\lambda_{m+1} = \inf_{\substack{\int u u_j = 0 \\ j=1,\cdots,m \\ u \not\equiv 0}} \frac{\int |\nabla u|^2}{\int u^2}$$

Thus,

$$\lambda_{m+1} \leq \frac{\int_\Omega |\nabla w_m|^2 \, \mathrm{d}x}{\int_\Omega w_m^2 \, \mathrm{d}x}$$

Also for $k \in \{1, \cdots, m\}$

$$\int_\Omega \nabla w_m \cdot \nabla u_k \, \mathrm{d}x \overset{IBP}{=} -\int_\Omega w_m \Delta u_k \, \mathrm{d}x = \lambda_k \int_\Omega w_m u_k \, \mathrm{d}x = 0$$

$$\implies \int_\Omega \nabla w_m \cdot \nabla u_k \, \mathrm{d}x = 0$$

$$\implies \int_\Omega \nabla w_m \cdot \nabla v_m \, \mathrm{d}x = 0$$

Since $\nabla v_m = \sum_{j=1}^{m} c_j \nabla u_j$.

$$\int_\Omega |\nabla v|^2 \, \mathrm{d}x = \int_\Omega |\nabla v_m + \nabla w_m|^2 \, \mathrm{d}x =$$

$$\int_\Omega |\nabla v_m|^2 \, \mathrm{d}x + 2 \int_\Omega \nabla v_m \cdot \nabla w_m \, \mathrm{d}x + \int_\Omega |\nabla w_m|^2 \, \mathrm{d}x$$

$$\implies \int_\Omega |\nabla w_m|^2 \,\mathrm{d}x \leq \int_\Omega |\nabla v|^2 \,\mathrm{d}x$$

$$\int_\Omega w_m^2 \,\mathrm{d}x \leq \frac{1}{\lambda_{m+1}} \int_\Omega |\nabla w_m|^2 \,\mathrm{d}x \leq \frac{1}{\lambda_{m+1}} \int_\Omega |\nabla v|^2 \,\mathrm{d}x$$

$m_2 \to \infty \implies \|w_m\|_{L^2} \to 0.$ □

Remark. Can do the same procedure to the Neumann eigenvalues and eigenfunctions. Only difference:

$$\lambda_1 = \inf_{\substack{u \in H^1(\Omega) \\ u \not\equiv 0}} \frac{\int_\Omega |\nabla u|^2 \,\mathrm{d}x}{\int_\Omega u^2 \,\mathrm{d}x}$$

$\lambda_1 = 0, u_1 = 1.$
$-\Delta u = \lambda_1 u$ in Ω.
$\nabla u \cdot \nu = 0$ on $\partial\Omega$.

$$\lambda_2 = \inf_{\substack{u \in H^1(\Omega) \\ \int_\Omega u \cdot 1 \,\mathrm{d}x = 0}} \mathcal{R}(u)$$

Proposition 3. Let $\lambda_1(\Omega) := \inf_{\substack{u \in H_0^1(\Omega) \\ u \not\equiv 0}} \mathcal{R}(u)$.
Then if $\Omega_1 \subset \Omega_2$ both open one has $\lambda_1(\Omega_2) \leq \lambda_2(\Omega_1)$.

Proof. Let u be any admissible function for $\lambda_1(\Omega_1)$. $u \in H_0^1(\Omega)$ $u \not\equiv 0$ Define completion for $\lambda_1(\Omega_2)$ via:

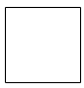
$$\widetilde{u}(x) = \begin{cases} u, & \text{if } x \in \Omega_1; \\ 0, & \text{if } x \in \Omega_2 \setminus \Omega_1. \end{cases}$$

Then $\widetilde{u} \in H_0^1$. But $\mathcal{R}_{\Omega_2}(\widetilde{u}) = \mathcal{R}_{\Omega_1}(u)$.
$\implies \lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$.

□

Example of 2D eigenfunction

Take $\Omega = (0,1) \times (0,1)$.



$-\Delta u = \lambda u$ in Ω.
$u = 0$ on $\partial\Omega$
We use separation of variables.
Seek $u(x,y) = F(x)G(y)$. Substitute into the PDE.

$$-F''(x)G(y) - F(x)G''(y) = \lambda F(x)G(y)$$

$$-\frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)} + \lambda \equiv \mu$$

$$F''(x) + \mu F(x) = 0$$

$$G''(y) + (\lambda - \mu)G(y) = 0$$

Then $F(0) = 0 = F(1)$ so $F(x) = \sin(\sqrt{\mu}x)$.
$F(1) = 0 \implies \sqrt{\mu} = k\pi \implies \mu = k^2\pi^2$

For the second case, if $\lambda - \mu$ is negative we have exponential but we also need $G(0) = G(1) = 0$ which is not possible. So it is not negative.

$$G(y) = \sin\Big(\sqrt{\lambda - \mu} y\Big)$$

$G(1) = 0 \implies \sqrt{\lambda - \mu} = l\pi$
$\implies \lambda = \mu + l^2\pi^2.$

$$\implies \lambda_{k,l} = (k^2 + l^2)\pi^2$$

where $k, l = 1, 2, 3, \cdots$.
Smallest eigenvalue is $\lambda_{1,1} = 2\pi^2$. Multiplicity is 1.
But for higher ones, we can have multiplicity > 1. $\lambda_{1,2} = \lambda_{2,1}$ with different eigenfunctions.

Monday, 11/4/2024

Recall that $\lambda_1 := \inf_{u \in H_0^1(\Omega), u \not\equiv 0} \mathcal{R}(u)$ for Dirichlet Boundary Condition.
For Neumann, we minimize just over H^1.
Recall from example of $\Omega = (0,1) \times (0,1)$:

$$\lambda_{k,l} = \pi^2(k^2 + l^2)$$

eigenfunctions were $u_{k,l}(x, y) = \sin(\pi k x)\sin(\pi l y)$
Then $\lambda_{1,2} = \lambda_{2,1}$ multiplicity 2 eigenvalue.

Proposition 4. Any first Dirichlet eigenfunction does not vanish in Ω.

Proof. Let u_1 be any first eigenfunction. Then, $\lambda_1 = \mathcal{R}(u_1)$.
$|u_1| \in H_0^1$.
$R(|u_1|) = \frac{\int_\Omega |\nabla |u_1|^2| \,\mathrm{d}x}{\int_\Omega |u_1|^2 \,\mathrm{d}x} = \mathcal{R}(u_1) = \lambda_1$.
Thus, $|u_1|$ is also a first eigenfunction.
Apply sobolev elliptic regularity: $|u_1|$ is smooth.
Then, $-\Delta(|u_1|) = \lambda_1 |u_1|$, $|u_1| = 0$ on $\partial\Omega$.
Then $\Delta(|u_1|) \le 0$.
This is a superharmonic function.
Strong Minimum Principle $\implies$ $|u_1|$ cannot achieve minimum inside Ω.
Thus, u_1 has no interior zeroes. □

Corollary 1. Every eigenfunction for λ_k with $k > 1$ must vanish somewhere

Proposition 5. λ_1 is always simple (multiplicity 1).

Proof. Suppose u_1 and $\widetilde{u}_1$ are linearly independent first eigenfunctions. WLOG, $u_1 > 0, \widetilde{u}_1 > 0$ in Ω and normalized so that $\int_\Omega u_1 \,\mathrm{d}x = \int_\Omega \widetilde{u}_1 \,\mathrm{d}x = 1$.
Let $w = u_1 - \widetilde{u}_1$. Then $\int_\Omega w \,\mathrm{d}x = 0$.
$-\Delta w = \lambda_1 w$ in Ω and $w = 0$ in $\partial\Omega$.
But $\nexists$ non-zero first eigenfunction that vanishes inside Ω. Contradiction.

□

Schauder Theory

Ref: Gilberg-Trudinger, J.Jost
Hölder spaces:

Definition 21 (Hölder Quotient)**.** for $\alpha \in (0,1)$:

$$[u]_{C^\alpha(\Omega)} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

This is not a norm. We define a norm by:

$$|u|_{C^{0,\alpha}} := \sup_{\Omega} |u(x)| + [u]_{C^{\alpha}(x)}$$

$$|u|_{C^{k,\alpha}(\Omega)} := \sum_{|\beta|\le k} \sup_{\Omega} |D^{\beta} u(x)| + \sum_{|\beta|=k} [D^{\beta} u]_{C^{\alpha}}$$

There are 2 sets of estimates in Schauder theory:

1) Very precise Hölder estimates for $\Delta u = f$ when $f \in C^{\alpha}$.

2) Use these to obtain similar estimates for $Lu = f$ where $L =$ general elliptic operator.

Recall:
Fundamental solution to Laplace's equation in n dim:

$$K(x) := \begin{cases} \dfrac{1}{n(2-n)\alpha_n} |x|^{2-n}, & \text{if } n > 2; \\ \dfrac{1}{2\pi} \ln |x|, & \text{if } n = 2. \end{cases}$$

$\alpha_n =$ volume of unit ball in $\mathbb{R}^n$.
[Evans had negative]
$K(x)$ is just a radial solution to $\Delta u = 0$.
K is <u>singular</u> at $x = 0$.
<u>Newtonian Potential</u> of a function $f : \Omega \to \mathbb{R}$ given by:

$$w(x) := \int_{\Omega} K(x-y) f(y) \,\mathrm{d}y$$

For nice enough f we can 'differentiate under the integral sign'.
Ignoring constants, $|K(x)| \le |x|^{2-n}$ or $|K(x)| \le \ln |x|$

$$\left| \frac{\partial K}{\partial x_i} \right| = \left| |x|^{1-n} \frac{x_i}{|x|} \right| \le |x|^{1-n}$$

$$\left| \frac{\partial^2 K}{\partial x_i \partial x_j} \right| = \left| |x|^{-n} \delta_{ij} + x_i(-n)|x|^{-n-1} \frac{x_i}{|x|} \right| = \left| \frac{\delta_{ij}}{|x|^n} - \frac{n x_i x_j}{|x|^{n+2}} \right| \le |x|^{-n}$$

For all n $K(x)$ is locally integrable.

$$\int_{B(0,1)} |x|^{2-n} \,\mathrm{d}x = \int_0^1 \int_{\partial B(0,r)} r^{2-n} \,\mathrm{d}S \,\mathrm{d}r = \int_0^1 r^{2-n} \int_{\partial B(0,r)} 1 \,\mathrm{d}S \,\mathrm{d}r$$

$$= \int_0^1 r^{2-n} n\alpha_n r^{n-1} \,\mathrm{d}r = \int_0^1 n\alpha_n r \,\mathrm{d}r < \infty$$

Also, $\frac{\partial K}{\partial x_i}$ is locally integrable.

$$\int_{B(0,1)} \left| \frac{\partial K}{\partial x_i} \right| \mathrm{d}x \sim n\alpha_n \int_0^1 r^{1-n} r^{n-1} \,\mathrm{d}r < \infty$$

However,

$$\int_{B(0,1)} \left| \frac{\partial^2 K}{\partial x_i \partial x_j} \right| \mathrm{d}x \sim \int_0^1 r^{-n} r^{n-1} \,\mathrm{d}r \sim \text{ logarithmic singularity}$$

Wednesday, 11/6/2024

We have the following basic estimates:

$$|K(x)| \leq C|x|^{2-n} \; (n > 2)$$

$$|K_{x_i}(x)| \leq C|x|^{1-n}$$

$$|K_{x_i x_j}| \leq C|x|^{-n}$$

Newtonioan Potential of function $f : \Omega \to \mathbb{R}$ is given by:

$$w(x) \coloneqq \int_\Omega K(x-y) f(y) \, \mathrm{d}x$$

Theorem 33 (A)**.** Assume f integrable and bounded.
Then w is C^1 on Ω and:

$$w_{x_i}(x) = \int_\Omega K_{x_i}(x-y) f(y) \, \mathrm{d}y$$

Proof. Let η be C^1 so that $\eta(x) = 0$ for $0 \leq x \leq 1$, $\eta(x) = 1$ for $x \geq 2$ and $0 \leq \eta' \leq 2$ for all x.
For $\epsilon > 0$ let $\eta_\epsilon(x) \coloneqq \eta\left(\frac{|x|}{\epsilon}\right)$.
Define

$$w_\epsilon(x) \coloneqq \int_\Omega K(x-y)\eta_\epsilon(x-y) f(y) \, \mathrm{d}y$$

Easy to justify:

$$w_{\epsilon_{x_i}}(x) = \int_\Omega \frac{\partial}{\partial x_i} \left(K(x-y)\eta_\epsilon(x-y)\right) f(y) \, \mathrm{d}y$$

Also easiy: $w_\epsilon \to w$ uniformly in Ω.

$$w - w_\epsilon = \int_\Omega K(x-y)(1-\eta_\epsilon(x-y)) f(y) \, \mathrm{d}y$$

$$= \int_{\{y : |x-y| < 2\epsilon\}} K(x-y)(1-\eta_\epsilon(x-y)) f(y) \, \mathrm{d}y$$

Now consider:

$$w_{\epsilon_{x_i}} - \int_\Omega K_{x_i}(x-y) f(y) \, \mathrm{d}y$$

$$= \int_\Omega \left(\frac{\partial}{\partial x_i}\left(K(x-y)\eta_\epsilon(x-y)\right) - K_{x_i}\right) f(y) \, \mathrm{d}y$$

$$= \underbrace{\int_\Omega K_{x_i}(x-y)(\eta_\epsilon(x-y) - 1) f(y) \, \mathrm{d}y}_{=\mathrm{I}}$$

$$+ \underbrace{\int_\Omega K(x-y)\eta_{\epsilon_{x_i}}(x-y) f(y) \, \mathrm{d}y}_{=\mathrm{II}}$$

$$\mathrm{I} \leq C \int_{\{y : |y-x| < 2\epsilon\}} \frac{1}{|x-y|^{n-1}} |f(y)| \, \mathrm{d}y \sim c \int_0^{2\epsilon} \frac{1}{r^{n-1}} r^{n-1} \, \mathrm{d}r = \mathcal{O}(\epsilon)$$

$$\mathrm{II} \leq \frac{2}{\epsilon} C \int_{\{y : \epsilon < |x-y| < 2\epsilon\}} \frac{1}{|x-y|^{n-2}} |f(y)| \, \mathrm{d}y = \mathcal{O}(\epsilon)$$

$$w(x+he_i) - w(x) = \lim_{\epsilon\to 0} w_\epsilon(x+he_i) - w_\epsilon(x) = \lim_{\epsilon\to 0}\int_0^h w_{\epsilon_{x_i}}(x+te_i)\,\mathrm{d}t$$

$$= \int_0^h \int_\Omega K_{x_i}(x+te_i-y)f(y)\,\mathrm{d}y$$

Divide by h:

$$\frac{w(x+he_i)-w(x)}{h} = \frac{1}{h}\int_0^h \int_\Omega K_{x_i}(x+te_i-y)f(y)\,\mathrm{d}y$$

Let $h\to 0$ to see $w_{x_i}(x) = \int_\Omega K_{x_i}(x-y)f(y)\,\mathrm{d}y$.

□

Theorem 34 (B)**.** Let $\Omega\subset\mathbb{R}^n$ be open bounded. Let $f\in C^{0,\alpha}(\Omega)$.
Then $w\in C^2(\Omega), \Delta w = f$ and $\forall x\in\Omega$ one has:

$$w_{x_ix_j} = \int_{\Omega_0} K_{x_ix_j}(x-y)(f(y)-f(x))\,\mathrm{d}y - f(x)\int_{\partial\Omega_0} K_{x_i}(x-y)\nu_j(y)\,\mathrm{d}S\,(*)$$

where Ω_0 is any smooth bounded open set such that $\Omega\subset\subset\Omega_0$ and ν is the outer unit normal to Ω_0 and $f\equiv 0$ outside Ω.

Proof. Let $u(x)$ be RHS of $(*)$.
Claim: $u(x)$ is well defined.
True since:

$$\int_{\Omega_0} |K_{x_ix_j}(x-y)||f(y)-f(x)|\,\mathrm{d}y$$

for $y\in\Omega$,

$$\le [f]_{C^\alpha}\int \frac{C}{|x-y|^n}|x-y|^\alpha\,\mathrm{d}y$$

$$\sim \int_0^\delta \frac{1}{r^n}r^{n-1}r^\alpha\,\mathrm{d}r \sim \delta^\alpha$$

For $y\notin\Omega$ we have $|x-y|>0$.
This proves the claim.
Fix i. Let $v_\epsilon(x) := \int_\Omega K_{x_i}(x-y)\eta_\epsilon(x-y)f(y)\,\mathrm{d}y$.
Theorem A $\implies$ $v_\epsilon\to w_{x_i}$ uniformly as $\epsilon\to 0$.
For any j:

$$v_{\epsilon_{x_j}}(x) = \int_\Omega \frac{\partial}{\partial x_j}\left(K_{x_i}(x-y)\eta_\epsilon(x-y)\right)f(y)\,\mathrm{d}y$$

$$= \int_{\Omega_0} \frac{\partial}{\partial x_j}\left(K_{x_i}(x-y)\eta_\epsilon(x-y)\right)f(y)\,\mathrm{d}y$$

$$= \int_{\Omega_0} \frac{\partial}{\partial x_j}\left(K_{x_i}(x-y)\eta_\epsilon(x-y)\right)(f(y)-f(x))\,\mathrm{d}y$$

$$+f(x)\int_{\Omega_0}\frac{\partial}{\partial x_j}(K_{x_i}(x-y)\eta_\epsilon(x-y))\,\mathrm{d}y$$

$$= \int_{\Omega_0} \frac{\partial}{\partial x_j}\left(K_{x_i}(x-y)\eta_\epsilon(x-y)\right)(f(y)-f(x))\,\mathrm{d}y$$

$$-f(x)\int_{\Omega_0}\frac{\partial}{\partial y_j}(K_{x_i}(x-y)\eta_\epsilon(x-y))\,\mathrm{d}y$$

$$= \int_{\Omega_0} \frac{\partial}{\partial x_j}\left(K_{x_i}(x-y)\eta_\epsilon(x-y)\right)(f(y)-f(x))\,\mathrm{d}y$$

$$-f(x)\int_{\partial\Omega_0}(K_{x_i}(x-y)\underbrace{\eta_\epsilon(x-y)}_{=1})\nu_j\,\mathrm{d}y$$

$$=\int_{\Omega_0}\frac{\partial}{\partial x_j}\left(K_{x_i}(x-y)\eta_\epsilon(x-y)\right)(f(y)-f(x))\,\mathrm{d}y$$

$$-f(x)\int_{\partial\Omega_0}K_{x_i}(x-y)\nu_j\,\mathrm{d}y$$

Then,

$$\left|u(x)-v_{\epsilon_{x_j}}(x)\right|=\int_{\{|y-x|<2\epsilon\}}\left|K_{x_ix_j}(x-y)\right||f(y)-f(x)|\,\mathrm{d}y$$

$$+\int_{\{\epsilon<|y-x|<2\epsilon\}}\underbrace{|\eta'_\epsilon|}_{\le\frac{2}{\epsilon}}|K_{x_i}(x-y)||f(y)-f(x)|\,\mathrm{d}y$$

$$\sim[f]_{C^\alpha}\int_0^{2\epsilon}\frac{1}{r^n}r^\alpha r^{n-1}\,\mathrm{d}r+\frac{2}{\epsilon}\int_\epsilon^{2\epsilon}\frac{1}{r^{n-1}}r^\alpha r^{n-1}\,\mathrm{d}r=\mathcal{O}(\epsilon^\alpha)$$

Thus $v_{\epsilon_{x_j}}\to u(x)$ uniformly as $\epsilon\to 0$.
As in theorem A, $w_{x_i}(x+he_j)-w_{x_i}(x)=\cdots$ and we're done.
Now we prove $\Delta w=f$.
Fixing x, take $\Omega_0=B(x,R)$ for large R [so that $\Omega\subset B(x,R)$]. Use $(*)$ with $i=j$, summing on i. This gives us the laplacian.

$$\Delta w=\sum_i w_{x_ix_i}=\underbrace{\int_{B(x,R)}\Delta K(x-y)(f(y)-f(x))\,\mathrm{d}y}_{0\text{ since }K\text{ is harmonic}}$$

$$+f(x)\int_{\partial B(x,R)}\nabla K(x-y)\cdot\nu\,\mathrm{d}S$$

Also,

$$\int_{\partial B(x,R)}\nabla K\cdot\nu\,\mathrm{d}S=\int_{\partial B(x,R)}\frac{\partial K}{\partial r}\,\mathrm{d}S=\int_{\partial B(x,R)}\frac{1}{n\alpha(n)R^{n-1}}\,\mathrm{d}S=1$$

So we're done.

□

Friday, 11/8/2024

Recap:

$$K(x)=\begin{cases}\dfrac{1}{\alpha_n n(2-n)}|x|^{2-n}, & \text{if } n>2;\\ \dfrac{1}{2\pi}\log|x|, & \text{if } n=2.\end{cases}$$

Newtonian potential of f in Ω:

$$w(x)=\int_\Omega K(x-y)f(y)\,\mathrm{d}y$$

So far: if $f\in C^{0,\alpha}(\Omega), 0<\alpha<1$ then $w\in C^2, \Delta w=f$ in Ω. We have estimate of derivative:

$$w_{x_ix_j}(x)=\int_{\Omega_0}K_{x_ix_j}(x-y)(f(y)-f(x))\,\mathrm{d}y-f(x)\int_{\partial\Omega_0}K_{x_i}(x-y)\nu_j(y)\,\mathrm{d}S$$

Where $\Omega_0 \supset\supset \Omega, \partial\Omega_0$ smooth, f is extended to be 0 on $\Omega_0 \setminus \Omega$
Note that if f is just continuous then $w \in C^2$.
This result, among other things, shows existence to $\Delta u = f$ in $\Omega, u = g$ on $\partial\Omega$ for $f \in C^{0,\alpha}, g \in C^0$. To show existence, it suffices to be able to solve $\Delta v = 0$ on Ω and $v = h$ on $\partial\Omega$ for all h continuous $(*)$.
Why?
Let $w = \int_\Omega K(x-y)f(y)\,\mathrm{d}y$
We seek u of the form:

$$u = w + \widetilde{u}$$

$$\Delta\widetilde{u} = 0 \text{ in } \Omega$$

$$\widetilde{u} = g - w \text{ on } \partial\Omega$$

How to solve $(*)$?
When $\Omega = B$ [a ball],

$$v = \frac{1}{\omega_{n-1}} \int_{\partial B_R} \frac{R - |x|^2}{|x-y|^n} h(y)\,\mathrm{d}S$$

So we have explicit Poisson Integral formula.
What if Ω is not a ball? Many methods:
Calculus of variations: minimize $\int_\Omega |\nabla v|^2\,\mathrm{d}x$
Perron's method.

Theorem 35. Let $B_R = B(x_0, R), B_{2R} = B(x_0, 2R)$ be 2 concentric balls in $\mathbb{R}^n$. Fix $R > 0$. Then for $f \in C^{0,\alpha}(B_{2R})$ we have

$$|w|_{C^{2,\alpha}(B_{2R})} \leq C(\alpha, R)|f|_{C^{0,\alpha}(B_{2R})}$$

Proof. To control $\sup|w_{x_ix_j}|$ in B_R:

$$|w_{x_ix_j}(x)| \leq \int_{B_{2R}} |K_{x_ix_j}| \frac{|f(y)-f(x)|}{|y-x|^\alpha} |y-x|^\alpha\,\mathrm{d}y + \left| f(x) \int_{\partial B_{2R}} K_{x_i}(x-y)\nu_j(y)\,\mathrm{d}S \right|$$

$$\leq \int_{B_{2R}} |K_{x_ix_j}| \frac{|f(y)-f(x)|}{|y-x|^\alpha} |y-x|^\alpha\,\mathrm{d}y + \sup_{B_{2R}}|f| \frac{C}{R^{n-1}} n\alpha(n)R^{n-1}$$

$$\leq C[f]_{C^\alpha} \int_0^{2R} \frac{1}{r^n} r^\alpha r^{n-1}\,\mathrm{d}r < C(R)[f]_{C^\alpha} + C\sup|f|$$

Then we bound $[w_{x_ix_j}]_{C^\alpha(B_R)}$.
Fix x and $\overline{x}$ in B_R.

$$w_{x_ix_j}(\overline{x}) - w_{x_ix_j}(x)$$

$$= \int_{B_{2R}} \left(K_{x_ix_j}(\overline{x}-y)(f(y)-f(\overline{x})) - K_{x_ix_j}(x-y)(f(y)-f(x)) \right)\,\mathrm{d}y$$

$$\underbrace{-f(\overline{x}) \int_{\partial B_{2R}} K_{x_i}(\overline{x}-y)\nu_j\,\mathrm{d}S_y + f(x) \int_{\partial B_{2R}} K_{x_i}(x-y)\nu_j\,\mathrm{d}S_y}_{=I_1+I_2+I_3+I_4}$$

$$I_1 = f(x) \int_{\partial B_{2R}} \left(K_{x_i}(x-y) - K_{x_i}(\overline{x}-y) \right) \nu_j\,\mathrm{d}S_y$$

$$+(f(x)-f(\overline{x})) \int_{\partial B_{2R}} K_{x_i}(\overline{x}-y)\nu_j\,\mathrm{d}S$$

$$\implies |I_1| \leq \sup_{B_{2R}} |f| \int_{\partial B_{2R}} |D_\tau K_{x_i}(\widetilde{x} - y)| \, |x - \overline{x}| \, \mathrm{d}S_y$$

$$+[f]_{C^\alpha} |x - \overline{x}|^\alpha \cdot C(R)$$

for some $\widetilde{x}$ between x and $\overline{x}$. D_τ is the directional derivative, we have used Mean Value Theorem.

$$\leq c \sup |f| \cdot \frac{1}{R^n} n\alpha(n)(2R)^{n-1} |x - \overline{x}| + C(R)[f]_{C^\alpha} |x - \overline{x}|^\alpha$$

$$\leq C(R)|f|_{C^{0,\alpha}} |x - \overline{x}|^\alpha$$

Let $\delta = |x - \overline{x}|$ and $\xi = \frac{1}{2}(x + \overline{x})$.

$$I_2 = \int_{B(\xi,\delta) \cap B_{2R}} K_{x_i x_j}(x - y)(f(x) - f(y)) \, \mathrm{d}y$$

$$+ \int_{B(\xi,\delta) \cap B_{2R}} K_{x_i x_j}(\overline{x} - y)(f(y) - f(x)) \, \mathrm{d}y$$

Estimate first integral:

$$\leq C[f]_{C^\alpha} \int_{B(\xi,\delta) \cap B_{2R}} \frac{1}{|x - y|^n} |x - y|^\alpha \, \mathrm{d}y$$

Notice that $B(x, \frac{3}{2}\delta) \supset B(\xi, \delta)$.

$$\leq C[f]_{C^\alpha} \int_{B(x, \frac{3}{2}\delta)} \frac{1}{|x - y|^n} |x - y|^\alpha \, \mathrm{d}y$$

$$\leq C[f]_{C^\alpha} \int_0^{\frac{3}{2}\delta} \frac{1}{r^n} r^\alpha r^{n-1} \, \mathrm{d}r \leq C[f]_{C^\alpha} \left(\frac{3}{2}\right)^\alpha \Bigg(\underbrace{\delta}_{|x - \overline{x}|}\Bigg)^\alpha$$

$$I_3 = (f(x) - f(\overline{x})) \int_{B_{2R} \setminus B(\xi,\delta)} K_{x_i x_j}(x - y) \, \mathrm{d}y$$

$$I_4 = \int_{B_{2R} \setminus B(\xi,\delta)} \left(K_{x_i x_j}(x - y) - K_{x_i x_j}(\overline{x} - y)\right) (f(\overline{x}) - f(y)) \, \mathrm{d}y$$

In I_3, we are 'outside' of the ball $B(\zeta, \delta)$ so $x - y$ doesn't get very small $|x - y| \geq \frac{\delta}{2}$. So, when we integrate we have enough control to win.
In I_4 we apply mean value theorem with the third derivative: $|D^3 K| \leq C|x|^{-n-1}$.
Dividing by $|x - \overline{x}^\alpha|$ we finish the proof.

□

Monday, 11/11/2024

Last estimates on $\Delta u = f$:
Last time:
If $f \in C^{0,\alpha}(B_{2R})$ and $\omega =$ Newtonian potential

$$\omega(x) = \int_{B_{2R}} K(x - y) f(y) \, \mathrm{d}y$$

then

$$|w|_{C^{2,\alpha}} \leq C(\alpha, R)|f|_{C^{0,\alpha}(B_{2R})}$$

Then suppose u is a $C^{2,\alpha}$ solution to $\Delta u = f$ in B_{2R}.
Then $u = w + v$

$$\Delta v = 0 \text{ in } B_{2R}$$

$$|u|_{C^{2,\alpha}(B_R)} \leq |w|_{C^{2,\alpha}(B_R)} + |v|_{C^{2,\alpha}(B_R)}$$

$$\implies |u|_{C^{2,\alpha}(B_R)} \leq C|f|_{C^{0,\alpha}(B_{2R})} + |v|_{C^{2,\alpha}(B_{2R})}$$

Recall derivtive estimates of harmonic functions:

$$|D^\beta v|_{B_R} \leq C(\beta, R) \sup_{B_{2R}} |v|$$

$\beta =$ any multi-index.

$$\implies |u|_{C^{2,\alpha}(B_R)} \leq C|f|_{C^{0,\alpha}(B_{2R})} + \sup_{B_{2R}} |u|$$

Interim Schauder estimate $\forall \Omega_0 \subset\subset \Omega$ bounded, open

$$\implies |u|_{C^{2,\alpha}(\Omega_0)} \leq C(\Omega_0, \alpha) \left(|f|_{C^{0,\alpha}(\Omega)} + \sup_{\Omega} |u| \right)$$

Cover Ω_0 with balls.

Theorem 36. [Boundary Hölder Estimates] Assume $u \in C^{2,\alpha}$ soluton to $\Delta u = f$ in $\Omega, u = g$ on $\partial\Omega$ where $f \in C^{0,\alpha}(\overline{\Omega}), g \in C^{2,\alpha}(\overline{\Omega})$ and $\partial\Omega \in C^{2,\alpha}$. Then,

$$|u|_{C^{2,\alpha}(\overline{\Omega})} \leq C \left(|f|_{C^{0,\alpha}(\overline{\Omega})} + |g|_{C^{2,\alpha}(\overline{\Omega})} + \sup_{\overline{\Omega}} |u| \right)$$

Philosophy: Flatten the boundary [we can do this since $\partial\Omega$ is $C^{2,\alpha}$].
Subtract off g to get zero dirichlet condition.
Then we work on $\frac{1}{2}$ balls.

General Schauder Estimates

Theorem 37. Let $Lu := a_{ij}u_{x_ix_j} + b_i(x)u_{x_i} + c(x)u, a_{ij} = a_{ji}, a_{ij}(x)\zeta_i\zeta_j \geq \lambda|\zeta|^2 \forall \zeta \in \mathbb{R}^n (\lambda > 0)$.

$$\sum_{i,j} |a_{ij}|_{C^{0,\alpha}} + \sum_i |b_i|_{C^{0,\alpha}} + |c|_{C^{0,\alpha}} \leq \Lambda$$

Proof. First step: Suppose a_{ij} is constant.
Idea: Change variables to convert $a_{ij}u_{x_ix_j}$ to Δu.
$Lu = a_{ij}u_{x_ix_j}$. a_{ij} is positive definite symmetric so there exists S orthogonal [$S^T = S^{-1}$] so that $S^{-1}AS = \widetilde{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ where $(A_{ij}) = a_{ij}$.
Let $y = xS$ where $x = (x_1, \cdots, x_n)$.
Let $v(y) = u(yS^{-1})$.
$u(x) = v(xS) \implies u_{x_i} = v_{y_k}S_{ik} \implies u_{x_ix_j} = v_{y_ky_l}S_{ik}S_{jl} \implies a_{ij}u_{x_ix_j} = v_{y_ky_l}S_{ik}a_{ij}S_{jl} = v_{y_ky_l}S^T_{ki}a_{ij}S_{jl} \implies a_{ij}u_{x_ix_j} = \widetilde{D}_{kl}v_{y_ky_l} = \widetilde{D}D^2v = \lambda_1 v_{y_1y_1} + \cdots \lambda_n v_{y_ny_n}$
Change variables again: let $z_i = \sqrt{\lambda_i}y_i$ then $\frac{\partial^2}{\partial z_i^2} = \frac{1}{\lambda_i}\frac{\partial^2}{\partial y_i^2} \implies w(z) = v(y) \implies \Delta_z w$.
If u solves $a_{ij}u_{x_ix_j} = f$ then w solves $\Delta_z = \widetilde{f}$.
Then we use old estimates for Poisson.

$\square$

Wednesday, 11/13/2024

We finish Schauder today.
$Lu := a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} + c(x)u$.
We assume uniform ellipticity: $a_{ij} = a_{ji}, a_{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2$ forsome $\theta > 0, \forall x \in \Omega$.
Also Hölder continuous coefficients.

$$\sum_{i,j} |a_{ij}|_{C^{0,\alpha}} + \sum_i |b_i|_{C^{0,\alpha}} + |c|_{C^{0,\alpha}}$$

So far, for $L = a_{ij}u_{x_ix_j}$ with a_{ij} constants, we showed that interior Hölder estimates work via change of variables.
$Lu = f, f \in C^{0,\alpha}(\Omega)$.
Then,

$$|u|_{C^{2,\alpha}(\Omega')} \leq C(\Omega, \Omega', \Lambda, \theta)(|f|_{C^{0,\alpha}(\Omega)} + \sup_\Omega |u|) \quad (*)$$

$\forall \Omega' \subset\subset \Omega$.

Theorem 38. If $Lu = f$ in $\Omega, f \in C^{0,\alpha}(\Omega)$ then $\forall \Omega' \subset\subset \Omega, (*)$ holds.

Proof. Let $x_0 \in \Omega'$. Rewrite $Lu = f$.

$$a_{ij}(x_0)u_{x_ix_j} = f - b_i(x)u_{x_i} - c(x)u + (a_{ij}(x_0) - a_{ij}(x))u_{x_ix_j}$$

$$=: F(x)$$

Now estimate $|F|_{C^{0,\alpha}}$ to use the previous lemma.
Take $B_R := B(x_0, R) \subset\subset \Omega$.
$|f|_{C^{0,\alpha}}$ is bounded from assumption.
We use the following fact: $[gh]_{C^\alpha} = \sup \frac{|g(x_2)(x_2) - g(x_1)h(x_1)|}{|x_2 - x_1|^\alpha}$
Same way as the product rule, we can write:

$$\leq \sup \frac{|g(x_2)||h(x_2) - h(x_1)|}{|x_2 - x_1|^\alpha} + \frac{|h(x_1)||g(x_2) - g(x_1)|}{|x_2 - x_1|^\alpha} \leq \sup |g|[h]_{C^\alpha} + \sup |h|[g]_{C^\alpha}$$

Then we can write:
$[b_iu_{x_i}]_{C^\alpha} \leq \sup |b_i|[u_{x_i}]_{C^\alpha} + \sup |u_{x_i}|[b_i]_{C^\alpha}$
$|b_iu_{x_i}|_{C^{0,\alpha}} \leq \Lambda|u|_{C^{1,\alpha}}$.
Similarly, $|cu|_{C^{0,\alpha}} \leq \Lambda|u|_{C^{0,\alpha}}$.
Finally,
$[(a_{ij}(x_0) - a_{ij}(x))u_{x_ix_j}]_{C^\alpha} \leq \sup |a_{ij}(x_0) - a_{ij}(x)|[D^2u]_{C^\alpha} + \Lambda \sup |D^2u|$
$\leq [a_{ij}]_{C^\alpha}R^\alpha[D^2u]_{C^\alpha} + \Lambda \sup |D^2u|$
This idea is called idea of freezing coefficient. We have variable coefficient, but we can take a ball in which the coefficients don't vary that much.
We have the following proposition:

Proposition 6. $\forall \epsilon \exists C_\epsilon > 0$ such that if $u \in C^{2,\alpha}(\Omega)$ then,

$$|u|_{C^2} \leq \epsilon|u|_{C^{2,\alpha}} + C_\epsilon \sup |u|.$$

Proof. Suppose not. Then we can find ϵ and $\{u_j\}$ such that

$$|u_j|_{C^2} > \epsilon|u_j|_{C^{2,\alpha}} + j \sup |u_j|$$

Setting $v_j := \frac{u_j}{|u_j|_{C^2}}$ we have same ineqaulity with $|v_j|_{C^2} = 1$.
We have $|v_j|_{C^{2,\alpha}} \leq \frac{1}{\epsilon}$.
Arzela-Ascoli $\implies v_{j_l} \xrightarrow{C^2} v$.
Then $|v|_{C^2} = 1$.
But $\sup |v_j| < \frac{1}{j}$ so $\sup |v| = 0$ so $v \equiv 0$.

∎

We use this estimate to finish the proof.
Applying this,

$$|F|_{C^{0,\alpha}} \leq |f|_{C^{0,\alpha}} + \epsilon|u|_{C^{2,\alpha}} + \Lambda R^{\alpha}|u|_{C^{2,\alpha}} + C_{\epsilon} \sup |u|$$

From previous lemma about constant coefficients,

$$|u|_{C^{2,\alpha}} \leq C\left(|f|_{C^{0,\alpha}} + \epsilon|u|_{C^{2,\alpha}} + \Lambda R^{\alpha}|u|_{C^{2,\alpha}} + C_{\epsilon} \sup |u|\right)$$

We pick R small enough so that $C; LR^{\alpha} \leq \frac{1}{2}$.
This choice just depends on $\Lambda, \theta, \Omega, \Omega'$. So we can cover Ω' with finitely many balls.

□

Boundary Schauder Estimate

Theorem 39. Let $\partial\Omega \in C^{2,\alpha}$ bounded, open. Assume $f \in C^{0,\alpha}(\overline{\Omega}), g \in C^{2,\alpha}(\overline{\Omega}), L$ as before. Assume u is a $C^{2,\alpha}$ solution to:

$$Lu = f \text{ in } \Omega$$

$$u = g \text{ on } \partial\Omega$$

Then $\exists C = C(\Omega, \Lambda, \theta)$ such that:

$$|u|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left(|f|_{C^{0,\alpha}} + |g|_{C^{2,\alpha}} + \sup_{\overline{\Omega}} |u|\right)$$

Proof. (Sketch) Subtract of g, so $u = g + v$ and v is 0 on $\partial\Omega$.
Then $Lv = f - Lg$.
Without any assumption on the sign of c, we need some norm of u on the RHS, since we can have eigenfunctions so $|u|$ can blow up as much as we want.
Proof works by flattening the boundary, working with half balls etc.

□

Friday, 11/15/2024

Last time:
$Lu = a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u$.
$a_{ij} = a_{ji}, a_{ij}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2$

$$|a_{ij}|_{C^{0,\alpha}(\overline{\Omega})} + |b_i|_{C^{0,\alpha}} + |c(x)|_{C^{0,\alpha}} < \Lambda$$

If $u \in C^{2,\alpha}(\overline{\Omega})$ solves

$$Lu = f \text{ in } \Omega, u = g \text{ on } \partial\Omega$$

($\partial\Omega \in C^{2,\alpha}, f \in C^{0,\alpha}, g \in C^{2,\alpha}(\overline{\Omega})$).
Then, $\exists K(\Omega, \Lambda, \theta)$

$$|u|_{C^{2,\alpha}(\overline{\Omega})} \leq K\left(|f|_{C^{0,\alpha}} + |g|_{C^{2,\alpha}} + \sup_{\overline{\Omega}} |u|\right)$$

Note: there might not always be a solution.

Theorem 40. Assume $c(x) \leq 0$ in Ω. Then we have $K(\Omega, \Lambda, \theta)$ such that

$$|u|_{C^{2,\alpha}(\overline{\Omega})} \leq K\left(|f|_{C^{0,\alpha}(\overline{\Omega})} + |g|_{C^{2,\alpha}(\overline{\Omega})}\right) \quad (*)$$

Proof. Recall if $c(x) \leq 0$ we saw that:

$$\sup_{\overline{\Omega}} u \leq \text{const}\left(\sup |f| + \sup_{\partial\Omega} |u|\right)$$

This allows us to eliminate $\sup_{\overline{\Omega}} u$ from the given inequality. □

Method of Continuity

Theorem 41. Under the previous assumptions (in particular $c(x) \leq 0$), $\exists$ a unique solution $u \in C^{2,\alpha}$ solving $Lu = f$ in $\Omega, u = g$ on $\partial\Omega$.

Proof. By subtracting off g $[u = v + g]$, WLOG we can assume 0 boundary condition. Then u solves $Lu = f$ on $\Omega, u = 0$ on $\partial\Omega$.
Then our inequality becomes:

$$|u|_{C^{2,\alpha}(\overline{\Omega})} \leq K|f|_{C^{0,\alpha}(\overline{\Omega})}$$

For each $t \in [0,1]$ we define:

$$L_t u := tLu + (1-t)\Delta u$$

Since L is elliptic, Δ is elliptic, L_t must also be elliptic. Directly, $(ta_{ij}(x) + (1-t)\delta_{ij})\zeta_i\zeta_j \geq (t\theta + (1-t))|\zeta|^2 \geq \min\{1,\theta\}|\zeta|^2 \ \forall \zeta \in \mathbb{R}$.
Also,

$$|a^t_{ij}, b_i, c| \leq \Lambda + 1$$

Thus means, since K depends on Ω, Λ, θ and we can choose same Λ, θ for all L_t, we have the same K for all L_t.
Goal: we want to show that $L_t u = f$ is solvable for $t = 1$.
We already have $L_0 u = f$ [simply $\Delta u = f$] is uniquely solvable with $(*) \iff |u|_{C^{2,\alpha}} \leq K|f|_{C^{0,\alpha}}$.
Let $B_1 := \{u \in C^{2,\alpha}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$.
$B_2 := C^{0,\alpha}(\overline{\Omega})$.
Then, for any t, $L_t : B_1 \to B_2$.
And also, $\|L_t u\|_{B_2} \leq C_1\|u\|_{B_1}$. Crucially, C_1 does not depend on t.
By $(*)$ we know that $\|u\|_{B_1} \leq K\|L_t u\|_{B_2}$.
Again, K independent of t.
Note that $(*) \implies \forall t, L_t$ is one-to-one since $L_t u_1 = L_t u_2 = f, u_1 = u_2 = 0$ on $\partial\Omega$ implies $\|u_1 - u_2\| \leq 0$.
Suppose for some $\tau \in [0,1]$ we know L_τ was onto. Then we can talk about the inverse:

$$L_\tau^{-1} : B_2 \to B_1$$

is well defined.
Then $\forall t \in [0,1], \forall f \in B_2$, $L_t u = f \iff L_\tau u = f + (L_\tau - L_t)u$
$= f + (\tau - t)Lu + [(1-\tau) - (1-t)]\Delta u = f + (t-\tau)(\Delta u - Lu)$
$\iff u = L\tau^{-1} f + (t-\tau)L_\tau^{-1}(\Delta - L)u =: Tu$.
Here $T : B_1 \to B_1$ and we seek a fixed point.
We want to apply <u>contraction mapping theorem</u>.
Let $u, v \in B_1$.

$$|Tu - Tv|_{B^1} = |t-\tau| \left|L_\tau^{-1}(\Delta - L)(u-v)\right|_{B^1}$$

$$\leq |t-\tau|K\left(|\Delta|_{B_1} + |L|_{B_1}\right)|u-v|_{B_1}$$

$$\leq |t-\tau|K \cdot 2C_1|u-v|_{B^1}$$

We can pick t close enough to τ so that $|t-\tau|K \cdot 2C_1 \leq \frac{1}{2}$ [just pick $|t-\tau| = \frac{1}{4KC_1}$]
Then T is a contraction mapping.
For $\tau = 0$ the laplacian is indeed 1-1, onto. Thus, there exists unique solution for $t \leq \frac{1}{4KC_1}$.
We can keep going like this until we reach 1 since K and C_1 doesn't depend on t or τ.

$\square$

Monday, 11/18/2024

Calculus of Variations

Basic problem:

$$\inf_{u\in\mathcal{A}} \underbrace{\int_\Omega L(x,u,\delta u)\,\mathrm{d}x}_{=:E(u)}$$

A minimizer will solve a PDE [typically nonlinear].
Roughly: a minimzer of E will have "derivative" 0. This property leads to a minimizer solving (weakly) a PDE known as the Euler-Lagrange Equation associated with E.
There are 2 notions of "derivative":
1. Gateaux derivative (generalization of a directional derivative).

Definition 22 (Gateaux Derivative)**.** Let $E : X \to \mathbb{R}$ where X is a Banach space. The Gateaux derivative of E at $u \in X$ denoted by $E'(u) \in X^*$ is defined through the property:

$$\forall h \in X : \lim_{t\to 0} \frac{1}{t}\left[E(u+th) - E(u) - \langle E'(u), th\rangle\right] = 0$$

To compute a Gateaux derivative, we have:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u+th)$$

2. Frechét Derivative (generalization of differentiable)

Definition 23 (Frechét derivative)**.** E is Frechét differentiable at u if $\exists E'(u) \in X^*$ such that:

$$\lim_{\substack{\|h\|\to 0\\ h\in X}} \frac{E(u+h) - E(u) - \langle E'(u), h\rangle}{\|h\|_X} = 0$$

Examples:

i) $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x \implies \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u+th) = \frac{1}{2}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} |\nabla u + t\nabla h|^2 = \frac{1}{2}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (|\nabla u|^2 + 2t\nabla u\cdot\nabla h + t^2|\nabla h|^2) = \nabla u\cdot\nabla h.$

ii) $E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{1}{p}\int_\Omega |\nabla u + t\nabla h|^p\,\mathrm{d}x = \left.\int_\Omega |\nabla u + t\nabla h|^{p-1}\frac{(\nabla u + t\nabla h)}{|\nabla u + t\nabla h|}\cdot\nabla h\,\mathrm{d}x\right|_{t=0}$$

$$= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla h\,\mathrm{d}x$$

Then u weakly solves:

$$-\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) = 0$$

iii) Plateau Problem / Soap Film Problem: $E(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x$, $u\big|_{\partial\Omega} = g$.

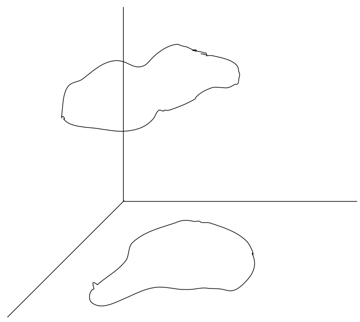


Figure 3:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u+th)$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_\Omega \sqrt{1+|\nabla u|^2 2t\nabla u\cdot\nabla h + t^2|\nabla h|^2}\,\mathrm{d}x = \int_\Omega \frac{\nabla u\cdot\nabla h}{\sqrt{1+|\nabla u|^2}}\,\mathrm{d}x$$

Thus a critical point u weakly solves:

$$\nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

This is called the minimal surface equation.

The LHS is called the mean curvature of the graph of u.

iv) Cahn-Hilliard Problem: $E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(u^2-1)^2\,\mathrm{d}x$. Model for phase transition. Also called Modica-Mortola problem.

$$\left(\inf_{\int_\Omega u\,\mathrm{d}x=0} E(u)\right)$$

forces a "phase transition".

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u+th) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_\Omega \frac{1}{2}|\nabla u + t\nabla h|^2 + \frac{1}{4}((u+th)^2-1)^2\,\mathrm{d}x$$

$$= \int_\Omega \nabla u\cdot\nabla h + \frac{1}{2}((u+th)^2-1)(2(u+th))h\,\mathrm{d}x$$

$$= \int_\Omega \nabla u\cdot\nabla h + (u^2-1)uh\,\mathrm{d}x$$

Weak form of $\Delta u = u^3 - u$, called the Allen-Cahn equation.

In general, $\Delta u = f(x,u)$ is called a semilinear Poisson equation.

Wednesday, 11/20/2024

Notation:
X = Banach Space
$E'(u) \in X^*$
Alternative notation: for $u, v \in X$ we have $\delta E(u; v)$ for $u, v \in X$ [alternative to $\langle E'(u), v \rangle$]
2nd variation: (Gateaux):

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} E(u + tv)$$

Notation: $\delta^2 E(u; v)$ for $u, v \in X$.
Proceding formally:
By 'Taylor's Theorem':

$$E(u + tv) \text{ `='} \; E(u) + \delta E(u; v) + \frac{1}{2}\delta^2 E(u; v)t^2 + \cdots$$

If u is a minimum, u is a critical point. Thus $\delta E(u; v) = 0 \forall t$.
Then, since $E(u) \leq E(u + tv)$ it follows that $\delta^2 E(u; v) \geq 0 \forall v$.
We have done this calculation 'formally'. Now we make it concrete.

Definition 24. If u is a critical point and $\delta^2 E(u; v) \geq 0 \forall v$ we say u is a stable critical point.
u is strictly stable if for some $v > 0$ we have:

$$\delta^2 E(u; v) \geq c|v|_X^2$$

$\forall v \in X$.

Proposition 7. A minimizer is stable.

Definition 25. We say u is a local minimizer of E if $\exists \delta > 0$ such that $E(u) \leq E(v)$ provided $|u - v|_X < \delta$.

Proposition 8. A local minimizer is stable.

Example 1. $E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + f(x, u)\,\mathrm{d}x$.

$$E(u + tv) = \int_\Omega \frac{1}{2}|\nabla u|^2 + t\nabla u \cdot \nabla v + \frac{t^2}{2}|\nabla v|^2 + f(x, u + tv)\,\mathrm{d}x$$

$$\delta E(u; v) = \int_\Omega \nabla u \cdot \nabla v + f_z(x, u)v\,\mathrm{d}x$$

Whee $f = f(x, z)$ with $z \in \mathbb{R}$.

$$\delta^2 E(u; v) = \int_\Omega |\nabla v|^2 + f_{zz}(x, u)v^2\,\mathrm{d}x$$

Suppose u is a critical point.

$$\delta E(u; v) = 0$$

$$\int_\Omega \nabla u \cdot \nabla v + f_z(x, u)v\,\mathrm{d}x = 0$$

$$\implies \int_\Omega (-\Delta u + f_z(x, u))\,v\,\mathrm{d}x = 0$$

This is true for all v so we need $\Delta u = f_z(x, u)$weakly.
Note that $f_{zz} \geq 0 \,\forall z$ is enough to gurantee that $\delta^2 E(u; v) \geq 0$.
Thus, if $z \mapsto f(x, z)$ convex for all x then u is stable.

Direct Method

Suppose we want to find:

$$m := \inf_{u \in \mathcal{A}} E(u)$$

Where:

$$E(u) = \int_\Omega L(x, u, \nabla u) \,\mathrm{d}x$$

Step 0: We want $m > -\infty$.
Consider a minimizing sequence $\{u_j\} \subset \mathcal{A}$ so that $E(u_j) \to m$.
Step 1: Compactness: Arguethat $u \in \mathcal{A}$ emerges as a limit of a subsequence of $\{u_j\}$.
Step 2: Lower Semi-Continuity (LSC):

$$\liminf_{j\to\infty} E(u_j) \geq E(u)$$

Then we have $E(u) \leq m \implies E(u) = m$.
Here, compactness generally comes from energy bound: $E(u_j) < m + 1 \forall j$.
We also perhaps have the fact that $u \in \mathcal{A}$.

Example 2. Suppose $E(u) = \int_\Omega |\nabla u|^2 + p(x)u^2 \,\mathrm{d}x$ where $p \geq 1$.
Then,

$$\int_\Omega |\nabla u_j|^2 + u_j^2 \,\mathrm{d}x < m + 1$$

So, $|u_j|_{H^1} < \sqrt{m+1}$
Kondrachov-Rellich implies:

$$u_{j_k} \xrightarrow{L^p} u \forall p < p^* = \frac{2n}{n-2}, n \geq 3$$

No info about ∇u_j.

$$u_{j_k} \overset{H^1}{\rightharpoonup} u \implies \int_\Omega \nabla u_{j_k} \cdot \nabla v \,\mathrm{d}x \to \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$$

Bounded sequence in a reflexive Banach space (here a Hilbert Space) are weakly compact.
The sobolev norm is weakly lower semi-continuous. Thus,

$$\liminf_{k\to\infty} \int_\Omega |\nabla u_{j_k}|^2 \,\mathrm{d}x \geq \int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

under weak H^1 convergence.
So ten we can solve:

$$\inf_{u \in H^1(\Omega)} \int_\Omega |\nabla u|^2 + p(x)u^2 \,\mathrm{d}x$$

with p smooth, $p(x) \geq 1$.
$m \geq 0$ ✓
$|u_j|_{H^1} < m + 1$
$u_{j_k} \overset{H^1}{\rightharpoonup} u$
$u_{j_k} \xrightarrow{L^p} u$ for $p < \frac{2n}{n-2}$

$$\liminf_{k\to\infty} \int_\Omega \left(|\nabla u_{j_k}|^2 + p(x)u_{j_k}^2\right) \,\mathrm{d}x$$

$$\geq \int_\Omega |\nabla u|^2 + p(x)u^2 \,\mathrm{d}x$$

Therefore u must be a minimum!

Friday, 11/22/2024

Some facts about weak convergence in a reflexive banach space X:

- $\|u_j\|_X < C \implies u_{j_k} \overset{X}{\rightharpoonup} u$
  $\ell \in X^*, \langle \ell, u_{j_k} \rangle \to \langle \ell, u \rangle$
- If $u_j \overset{X}{\rightharpoonup} u$ then $\liminf_{j\to\infty} \|u_j\|_X \geq \|u\|_X$
- Mazur's Lemma: If $K \subset X$ with K convex and closed under strong convergence then K is weakly closed.

  That is, if K convex and if $\forall \{u_j\} \subset K : u_j \overset{X}{\to} u \implies u \in K$,

  then $u_j \overset{X}{\rightharpoonup} u \implies u \in K$.

An example of weak convergence in L^2 :
Let $\rho(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ 2, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$
Extend periodically: $\rho(x+1) = \rho(x) \forall x \in \mathbb{R}$:
Define: $\rho_k(x) := \rho(kx), k = 1, 2, 3$ for $0 \leq x \leq 1$.
For ϕ smooth: we see:

$$\lim_{k\to\infty} \int_0^1 \rho_k(x)\phi(x)\,\mathrm{d}x = \frac{3}{2}\int_0^1 \phi(x)\,\mathrm{d}x$$

Then $\rho_k \overset{L^2}{\rightharpoonup} \frac{3}{2}$.

Example 3. Let $f(x,z)$ be continuous for $x \in \Omega, z \in \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ open, bounded and $0 \leq f(x,z) \forall x \in \Omega, \forall z \in \mathbb{R}$.
Let $E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + f(x,u)\,\mathrm{d}x$
Define $m := \inf_{u\in\mathcal{A}} E(u)$
Where $\mathcal{A} = \{u \in H^1(\Omega), u = g \text{ on } \partial\Omega\}$.
Assume $\exists G : \Omega \to \mathbb{R}$ such that $G \in H^1(\Omega)$ and $E(G) < \infty$ and $\operatorname{tr} G = g$ on $\partial\Omega$.
$E(G) < \infty$ condition implies $m < \infty$.
In direct method, we want to make sure our admissible set isn't empty!
Since $f \geq 0$ we also have $m \neq -\infty$ since $m \geq 0$.
Compactness:
$\{u_j\} \subset \mathcal{A}, E(u_j) \to m$.

$$\int_\Omega \frac{1}{2}|\nabla u_j|^2 + f(x,u_j)\,\mathrm{d}x < m+1 \implies \int_\Omega |\nabla u_j|^2\,\mathrm{d}x < 2m$$

Now consider $\{u_j - G\} \subset H_0^1(\Omega)$.
Poincaré $\implies$

$$\int_\Omega |u_j - G|^2\,\mathrm{d}x \leq C_p \int_\Omega |\nabla u_j - \nabla G|^2\,\mathrm{d}x \leq \text{const independent of } j$$

Thus, $\int_\Omega u_j^2\,\mathrm{d}x \leq$ const.
Thus, $\|u_j\|_{H^1} <$ const
$\implies u_{j_k} \overset{H^1}{\rightharpoonup} u_*$
Also, $u_{j_k} \overset{L^2}{\to} u_* \implies u_{j_{k_l}} \overset{\text{pointwise a.e.}}{\to} u_*$
for some $u_* \in H^1$
Lower semicontinuity:

$$\liminf_{k\to\infty} E(u_{j_{k_l}}) = \liminf_{k\to\infty} \int_\Omega \frac{1}{2}|\nabla u_{j_{k_l}}|^2 + f(x, u_{j_{k_l}})\,\mathrm{d}x$$

$$\geq \int_\Omega \frac{1}{2}|\nabla u_*|^2\,\mathrm{d}x + \liminf_{k\to\infty} \int_\Omega f(x, u_{j_{k_l}})\,\mathrm{d}x$$

$$\overset{\text{Fatou's Lemma}}{\Longrightarrow} f(x, u_{j_{k_l}}) \overset{\text{pointwise}}{\to} f(x, u_*)$$

f is continuous, so,

$$\geq \int_\Omega \frac{1}{2}|\nabla u_*|^2 + f(x, u_*) \, \mathrm{d}x = E(u_*)$$

pointwise a.e. means there's no gurantee of having desired result on the boundary (which is a measure 0 set).
Claim: $\mathcal{A}$ is strongly closed under strong H^1 convergence.
Also, $\mathcal{A}$ is convex.
Why closed? If $\{v_j\} \subset \mathcal{A}, v_j \overset{H^1}{\to} v \implies v_j - G \overset{H^1}{\to} v - G$.
What about the trace?

$$\int_{\partial\Omega} |v - G|^2 \, \mathrm{d}x \leq \int_{\partial\Omega} |v - v_j|^2 \, \mathrm{d}x + \underbrace{\int_{\partial\Omega} |v_j - G|^2 \, \mathrm{d}x}_{\to 0}$$

$$\leq C \int_\Omega |v - v_j|^2 + |\nabla v - \nabla v_j|^2 \, \mathrm{d}x \to 0$$

Why convex?
Let $v_1, v_2 \in \mathcal{A}$

$$\lambda v_1 + (1 - \lambda) v_2 - G \in H_0^1(\Omega)$$

$\lambda(v_1 - G) + (1 - \lambda)(v_2 - G) \in H_0^1(\Omega)$
By Mazur, $\mathcal{A}$ is weakly closued.
$u_* \in \mathcal{A}$.

Formally, u_* is a weak solution of

$$\Delta u_* = f_z(x, u_*), u_* = g \text{ on } \partial\Omega$$

So $f(x, z)$ being continuous is not enough.

Monday, 12/2/2024

What can go wrong with the direct method?

1) $m := \inf_{u \in \mathcal{A}} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - u^p\right) \, \mathrm{d}x$ with $p > 2, \mathcal{A} = H_0^1(\Omega)$. Here $m = -\infty$. To see this, consider $u_0 \in H_0^1(\Omega), u \not\equiv 0$. Calling the functional $E(u)$, we see that $E(u_0) = \int_\Omega \frac{1}{2}|\nabla u_0|^2 - u_0^p \, \mathrm{d}x$. Also, $E(\lambda u_0) = \int_\Omega \frac{\lambda}{2}|\nabla u_0|^2 - \lambda^p u_0^p \, \mathrm{d}x = \lambda^2 \int_\Omega \frac{1}{2}|\nabla u_0|^2 - \lambda^{p-2} u_0^p \, \mathrm{d}x$.

   $\lambda u_0 \in \mathcal{A}$ for all λ so we can send $\lambda \to \infty$ of $-\infty$ and $m = -\infty$.

   It is not all over. We might find local minimizers, we might find saddle points. But no global minimizer, so direct method might work.

2) $m := \inf_{u \in \mathcal{A}} \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x$ where $\mathcal{A} = \left\{u \in H_0^1(\Omega) : \int_\Omega |u|^p \, \mathrm{d}x = 1\right\}$

   We definitely have $m \geq 0$.

   What about compactness? If we take $\{u_j\} \subset \mathcal{A}$ to be a minimizing sequence then we can have $\frac{1}{2} \int_\Omega |\nabla u_j|^2 \, \mathrm{d}x < m + 1$ so we can have $\|u_j\|_{H_0^1} <$ const.

   Compactness implies $u_{j_k} \overset{H^1}{\rightharpoonup} u_0$ so by Rellich-Kondrachov we have $u_{j_k} \overset{L^q}{\to} u_0$ $\forall q < 2^* = \frac{2n}{n-2}$.

   What if $p \geq 2^*$?

   $\int_\Omega |u_j|^p \, \mathrm{d}x = 1 \, \forall j$.

   Lower semicontinuity implies $\liminf_{j \to \infty} \int_\Omega |\nabla u_j|^2 \, \mathrm{d}x \geq \int_\Omega |\nabla u_0|^2 \, \mathrm{d}x$.

   However, no gurantee that $\lim_{j \to \infty} \int_\Omega |u_j|^p \, \mathrm{d}x = \int_\Omega |u_0|^p \, \mathrm{d}x$.

3) $m := \inf_{u \in W^{1,4}(0,1)} \underbrace{\int_0^1 ((u')^2 - 1)^2 + u^2 \, \mathrm{d}x}_{=E(u)}$.

We have $m \geq 0$.

Let u_j be a minimizing sequence. Then $E(u_j) \to m$.

Compactness $\implies E(u_j) \leq m + 1$.

Then $\|u_j\|_{W^{1,4}} \leq$ const.

$u_{j_k} \overset{W^{1,4}}{\rightharpoonup} u_0 \implies u_{j_k} \overset{L^q}{\to} u_0$ for $q < 4^*$.

But we don't necessarily have lower semicontinuity.

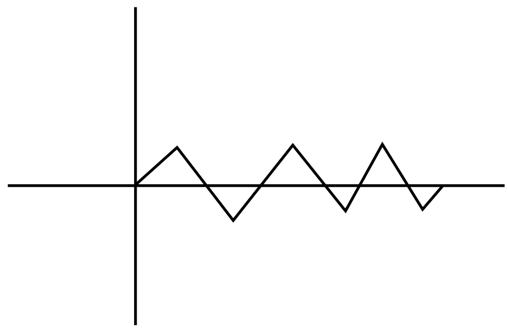


Figure 4:

here $u' = \pm 1$ so we don't have problem. Minimizing: we make it as small as ppossible.

$0 = \liminf_{j \to \infty} E(u_j) < E(0)$

$u_j \rightharpoonup u_0 = 0$

Let $\rho(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1; \\ 2 - x, & \text{if } 1 < x < 2. \end{cases}$

And periodic so $\rho(x + 2) = \rho(x)$.

Define $\rho_\epsilon(x) = \epsilon \rho(\frac{x}{\epsilon})$

Then $\rho_\epsilon \overset{H^1}{\rightharpoonup} 0$.

Note: $\rho_\epsilon \overset{L^2}{\to} 0, \rho'_\epsilon \overset{L^2}{\not\to} 0$

1 Notions of Convexity

1) A set D is <u>convex</u> if $p, q \in D \implies \lambda p + (1 - \lambda) q \in D \forall \lambda \in (0, 1)$.

2) Convex functions: let D be a convex set. Then $f : D \to R$ is convex if $f(\lambda p + (1 - \lambda) q) \leq \lambda f(p) + (1 - \lambda) f(q) \, \forall p, q \in D \forall \lambda \in (0, 1)$

Characterizations of convex functions:

Proposition 9. if f is twice differentiable on $D \subset \mathbb{R}^n$ where D is convex, then f convex if $D^2 f$ is positive definite, eg

$$\xi^T D^2 f(x) \xi > 0 \, \forall \xi \in \mathbb{R}^n \setminus \{0\} \forall x \in D$$

Proposition 10. If $f : D \to R$ is convex and differentiable then $\forall p, q \in D$

$$f(q) \geq f(p) + Df(p) \cdot (q - p)$$

Graph of f lies above every tangent plane!

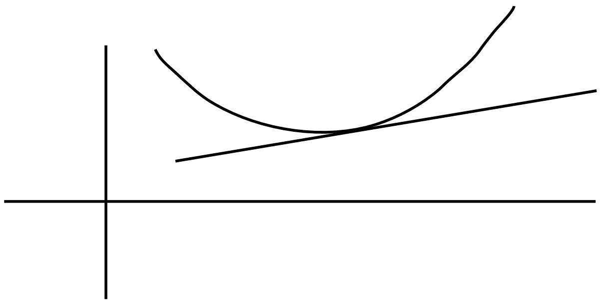


Figure 5:

Wednesday, 12/4/2024

One more troubling example where direct method seems to fail.
This is one of the most famous problems in calculus of variations!

$$m := \inf_{u \in \mathcal{A}} \int_\Omega \sqrt{1 + |\nabla u|^2} \, \mathrm{d}x$$

Where $\mathcal{A} = \left\{ u : u - G \in W_0^{1,1}(\Omega) : \right\}$ where $G \in W^{1,1}(\Omega)$ given [just way of phrasing the boundary conditions].
Let $\{u_j\}$ be a minimizing sequence. We know $-\infty < m \leq E(G) < \infty$. Then $E(u_j) \to m$.
Question: Do we have compactness?

$$\int_\Omega \sqrt{1 + |\nabla u_j|^2} \, \mathrm{d}x < m + 1$$

$$\implies \|\nabla u_j\|_{L^1} < m + 1$$

What about $\|u_j\|_{L^1}$? We can say $u_j - G \in W_0^{1,1}$ so Poincaré $\implies$ $\|u_j - G\|_{L^1} < \|\nabla u_j - \nabla G\|_{L^1} < \text{const} \implies \|u_j\|_{W^{1,1}} < \text{const}$.
L^1 is not reflexive. So we don't have any $W^{1,1}$ weakly convergent subsequence!
What aboout $W^{1,2}$? We don't have L^2 bound, so we can't control $\|u_j\|_{H^1}$ by $E(u_j)$!

Sufficient Conditions for Success of the Direct Method

Theorem 42. Let $\Omega \subset \mathbb{R}^n$, open, bounded. Consider the lagrangian $L = L(x, z, p)$ for $x \in \Omega, z \in \mathbb{R}, p \in \mathbb{R}^n$ be C^2 and convex in p, meaning for every $x \in \Omega, z \in \mathbb{R}$, the map $p \mapsto L(x, z, p)$ is convex. Also assume L is bounded below.
Then if $w_j \overset{W^{1,q}}{\rightharpoonup} u$ for some $1 < q < \infty$ one has lower semicontinuity:

$$\liminf_{j \to \infty} \int_\Omega L(x, u_j, \nabla u_j) \, \mathrm{d}x \geq \int_\Omega L(x, u, \nabla u) \, \mathrm{d}x$$

Example 4.

$$E(u) = \int_\Omega a_{ij}(x, u) u_{x_i} u_{x_j} \, \mathrm{d}x$$

Here assume matrhix $A(x, z)$ with entries a_{ij} is positive definite $\forall x \in \Omega, z \in \mathbb{R}^n$.
$L(x, z, p) = a_{ij}(x, z) p_i p_j \quad (*)$.

Proof. Special Case:
Assume $L = L(p)$ and $L(p) \geq 0$. Fix $\epsilon > 0$ and define $\Omega_\epsilon := \{x \in \Omega : |\nabla u| \leq 1/\epsilon\}$
Convexity $\implies \int_{\Omega_\epsilon} L(\nabla u_j)\,\mathrm{d}x \geq \int_{\Omega_\epsilon} L(\nabla u) + \nabla_p L(\nabla u)\cdot(\nabla u_j - \nabla u)\,\mathrm{d}x$.
On Ω_ϵ ,

$$|\nabla_p L(\nabla u)|_{L^\infty} \leq \sup_{|p|\leq 1/\epsilon} |\nabla_p L(p)| \leq C_\epsilon$$

Also,

$$\infty > \int_{\Omega\setminus\Omega_\epsilon} |\nabla u|^q\,\mathrm{d}x \geq \frac{1}{\epsilon^q}|\Omega \setminus \Omega_\epsilon|$$

$$\implies |\Omega \setminus \Omega_\epsilon| \to 0 \text{ as } \epsilon \to 0$$

Note:

$$\lim_{j\to\infty} \int_{\Omega_\epsilon} \nabla_p L(\nabla u)\cdot \nabla u_j\,\mathrm{d}x = \int_{\Omega_\epsilon} \nabla_p L(\nabla u)\cdot\nabla u\,\mathrm{d}x$$

$\nabla_p L(\nabla u)$ lives in L^∞ so in particular it lives in $L^{q'}$ where $\frac{1}{q} + \frac{1}{q'} = 1$.
By $(*)$ we have:

$$\underbrace{\liminf_{j\to\infty} E((u))}_{=\int_\Omega L(\nabla u)\,\mathrm{d}x} \geq \liminf_{j\to\infty} \int_{\Omega_\epsilon} L(\nabla u_j)\,\mathrm{d}x \geq \int_{\Omega_\epsilon} L(\nabla u)\,\mathrm{d}x = \int_\Omega \chi_{\Omega_\epsilon} L(\nabla u(x))\,\mathrm{d}x$$

Up to now, ϵ was fixed.

$$\chi_\epsilon(x) L(\nabla u(x)) \overset{\text{pointwise a.e.}}{\to} L(\nabla u(x))$$

Also monotonically. By monotone convergence theorem, setting $\epsilon \to 0$ we reach the result.

□

In fact, convexity is necessary as well. Suppose $\exists p_1, p_2 \in \mathbb{R}^n$ such that:

$$L(\lambda p_1 + (1-\lambda)p_2) > \lambda L(p_1) + (1-\lambda)L(p_2)$$

Same idea as oscillating between ± 1 we build u_j such that ∇u_j oscilates between p_1 and p_2.
Then, $\nabla u_j \overset{L^q}{\rightharpoonup} \overline{\lambda} p_1 + (1-\overline{\lambda})p_2 \equiv u_0$ for some $\overline{\lambda} \in (0,1)$. Then,

$$\liminf_{j\to\infty} \int_\Omega L(\nabla u_j)\,\mathrm{d}x < \int_\Omega L(\nabla u_0)\,\mathrm{d}x$$

Friday, 12/6/2024

Last time: For $1 < q < \infty$: $\forall (x,z) \in \Omega \times \mathbb{R}\, p \mapsto L(x,z,p)$ is convex
Iff $\forall u_j \overset{W^{1,q}}{\rightharpoonup} u$,

$$\liminf_{j\to\infty} \int_\Omega L(x, u_j, \nabla u_j)\,\mathrm{d}x \geq \int_\Omega L(x,u,\nabla u)\,\mathrm{d}x$$

We also had the theorem: Assuming the convexity condition, the following are necessary and sufficient for direct method to work:
Suppose $G \in W^{1,q}(\Omega)$ satisfies the Dirichlet condition.
Also, $E(G) = \int_\Omega L(x, G, \nabla G)\,\mathrm{d}x < \infty$.
Coercivity condition: $L((x,z,p)) \geq c_1|p|^q - c_2$.
Then $\exists$ minimizer for:

$$m = \inf_{u - G \in W_0^{1,q}(\Omega)} E(u)$$

Direct Method. Let $\{u_j\}$ be minimizing sequence, $E(u_j) \to m$.

$$-c_2 \leq c_1 \int_\Omega |\nabla u_j|^q - c_2 \,\mathrm{d}x \leq E(u_j)m \leq +1$$

$$\|\nabla u_j\|_{L^q} < \frac{c_2 + m + 1}{c_1}$$

$\|u_j - G\|_{L^q} \overset{\text{Poincarë}}{\leq} C_p \|\nabla u_j - \nabla G\|_{L^q} \leq \text{const.}$
$\implies \|u_j\|_{W^{1,q}} < \text{const}$
$\implies u_{j_k} \overset{W^{1,q}}{\rightharpoonup} u.$
Convexity $\implies$ weak lowersemicontinuity therefore $E(u) \leq \liminf_{j\to\infty} E(u_j) = m$

□

$\delta E(u; v) = 0 \,\forall v \in W_0^{1,q}$ with some extra assumptions.
Assuming we have these assumptions, we have (in the Gateaux sense):

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_\Omega L(x, u + tv, \nabla u + t\nabla v) \,\mathrm{d}x = 0$$

$$\implies \cdots \implies$$

$$-\underbrace{L_{p_i p_j}(x, u, \nabla u)}_{:=a_{ij}(x,u(x),\nabla u(x))} u_{x_i x_j} = L_z(x, u, \nabla u) + L_{p_i x_i} L(x, u, \nabla u) + L_{p_i z}(x, u, \nabla u) u_{x_i}$$

a_{ij} is positive definite from strict convexity. So this is elliptic! But it is nonlinear.

Theorem 43. Assume $L = L(x, p)$ [so not on ∇u] for $x \in \Omega, p \in \mathbb{R}^n$. Assume L is smooth. Assume $\theta > 0$ such that $L_{p_i p_j}(x, p)\xi_i \xi_j \geq \theta |\xi|^2 \,\forall \xi \in \mathbb{R}^n$ for all $x \in \Omega, \forall p \in \mathbb{R}^n$ [it is uniformly convex].
Then there exists at most one minimizer to

$$m := \inf_{u - G \in W_0^{1,q}(\Omega)} E(u) \text{ where } E(u) = \int_\Omega L(x, u, \nabla u) \,\mathrm{d}x$$

Proof. Suppose u_1, u_2 are both minimizers.
Then $E(u_1) = m = E(u_2)$.
Let $v := \frac{u_1 + u_2}{2}$. Note: v must be admissible since $v - G \in W_0^{1,q}$.
Taylor's theorem $\implies \forall p_1, p_2 \in \mathbb{R}^n, L(x, p_2) = L(x, p_1) + \nabla_p L(x, p_1) \cdot (p_2 - p_1) + \frac{1}{2}(p_2 - p_1)^T D^2 L(x, \widetilde{p})(p_2 - p_1)$.
Choose $p_1 = \nu v = \frac{\nabla u_1 + \nabla u_2}{2}$ and $p_2 = \nabla u_1$.
$E(u_1) = \int_\Omega L(x, \nabla u_1) \,\mathrm{d}x \geq \int_\Omega L(x, \nabla v) + \nabla_p L(x, \nabla v) \cdot \frac{\nabla u_1 - \nabla u_2}{2} \,\mathrm{d}x$
$+\frac{1}{2}\theta \int_\Omega \frac{|\nabla u_1 - \nabla u_2|^2}{4} \,\mathrm{d}x$
Similarly, $E(u_2) \geq E(v) + \int_\Omega \nabla_p L(x, \nabla v) \cdot \frac{\nabla u_2 - \nabla u_1}{2} \,\mathrm{d}x + \frac{\theta}{8} \int_\Omega |\nabla u_2 - \nabla u_1|^2 \,\mathrm{d}x$
Adding,
$2m \geq 2E(v) + \frac{\theta}{4} \int_\Omega |\nabla u_2 - \nabla u_1|^2 \,\mathrm{d}x \implies E(v) < m$ which is a contradiction.

□

An example of non-uniqueness:

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(u^2 - 1)^2 \,\mathrm{d}x$$

$m = \inf_{u \in H_0^1(\Omega)} E(u)$.
Euler-Lagrangian Equation: $\Delta u = u^3 - u$ in Ω and $u = 0$ on $\partial\Omega$.
Note: $u \equiv 0$ is a critical point!
Is it a minimizer? Sometimes it is, sometimes it isn't.
Let's compute the 2nd variation of E at $u \equiv 0$.

$$\delta^2 E(u;v) = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \int_\Omega \frac{1}{2}|\nabla u + t\nabla v|^2 + ((u+tv)^2-1)^2 \,\mathrm{d}x$$
$$= \int_\Omega |\nabla v|^2 + \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \cdots \,\mathrm{d}x$$

When $u \equiv 0$ we have:

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \frac{1}{4}\left((0+tv)^2-1\right)^2 = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \frac{1}{4}\left(t^4v^4 - 2t^2v^2+1\right) = -v^2$$

Therefore, $\delta^2 E(0;v) = \int_\Omega |\nabla v|^2 - v^2 \,\mathrm{d}x \geq \int_\Omega \lambda_1 v^2 - v^2 \,\mathrm{d}x$
Since $\lambda_1 \coloneqq \inf_{v \in H_0^1} \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega v^2 \,\mathrm{d}x}$ and exists due to Poincaré.
If we pick $v \coloneqq$ first eigenfunction of $-\Delta v_1 = \lambda_1 v_1, v_1 = 0$ on $\partial\Omega$.
Then, $\delta^2 E(0;v_1) = \int_\Omega (\lambda_1 - 1) v_1^2 \,\mathrm{d}x$
Then 0 will be unstable provided $\lambda_1 < 1$.
Therefore, 0 is definitely not the minimizer.
However, there definiely exists a minimizer u^* by the direct method.
Since $E(-u^*) = E(u^*)$, $-u^*$ must also be a minimizer! Thus there exsits at least 2 minimizers. So we have non-uniqueness of minimizers.

Monday, 12/9/2024

Final @ Wednesday 10:20-12:20, RH104

Constrained Variational Problems

General Problem:

$$\inf_{u \in \mathcal{A}} E(u)$$

$\mathcal{A}$ includes some constraints in addition to Sobolev and boundary condition, such as: $J(u) = 0, u(x) \lesseqgtr h(x), |\nabla u(x)| \leq 1$ etc.
How does a constraint affect the Euler Lagrange equation?
example: $\inf_{u \in \mathcal{A}} \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x$ where $\mathcal{A} = \{u \in H_0^1(\Omega) : J(u) = 0\}$ where $J(u) = \int_\Omega G(u(x)) \,\mathrm{d}x$.
We will assume $G : \mathbb{R} \to \mathbb{R}$ smooth. We assume the bound $|G(z)| \leq C(|z|^2 + 1)$.

1) A minimizer exists: $\{u_j\} =$ min. seq $\subset \mathcal{A}$.
   $\int_\Omega |\nabla u_j|^2 \,\mathrm{d}x \leq 2m + 1$.
   $\int_\Omega u_j^2 \,\mathrm{d}x \leq C_p \int_\Omega |\nabla u_j|^2 \,\mathrm{d}x$
   $\|u_j\|_{H^1} <$ const.
   $u_{j_k} \overset{H^1}{\rightharpoonup} u_0, u_{j_k} \overset{L^2}{\rightharpoonup} u_0$.
   $\liminf_{k \to \infty} \int_\Omega |\nabla u_{j_k}|^2 \,\mathrm{d}x \geq \int_\Omega |\nabla u_0|^2 \,\mathrm{d}x$.
   We need to verify if u_0 is admissible.
   Since $\int_\Omega G(u_j) \,\mathrm{d}x \leq \int_\Omega (c|u_j|^2 + 1) \,\mathrm{d}x$, by dominated convergence theorem we have $\int_\Omega G(u_0) \,\mathrm{d}x = 0$.
   Strategy: We want to find a differential equation that the minimizer must solve. We get Euler Lagrangian equation from differentiating the curve $t \mapsto u + tv$. But $u + tv$ is not admissible.
   We want to build a curve of competitors and differenitate along that curve to get a new criticality condition.
   Suppose $g = G'$.
   We <u>assume</u> that $|\{x \in \Omega : g(u_0(x)) \neq 0\}| > 0$.

We select any $w \in H_0^1(\Omega)$ such that $\int_\Omega g(u_0(x))w(x)\,\mathrm{d}x \neq 0$.

For this w, for $\tau, \sigma \in \mathbb{R}$ and any $v \in H_0^1(\Omega)$, define:

$$j(\tau, \sigma) := J(u_0 + \tau v + \sigma w)$$

We're looking for an admissible curve.

$j(0,0) = J(u_0)$ since $u_0 \in \mathcal{A}$.

We want to use Implicit Function Theorem.

$\frac{\partial j}{\partial \tau} = \frac{\partial}{\partial \tau} \int_\Omega G(u_0 + \tau v + \sigma w)\,\mathrm{d}x = \int_\Omega g(u_0 + \tau v + \sigma w)v\,\mathrm{d}x$

$\frac{\partial j}{\partial \sigma} = \int_\Omega g(u_0 + \tau v + \sigma w)w\,\mathrm{d}x$

Since $\int_\Omega g(u_0(x))w(x)\,\mathrm{d}x \neq 0$,

$\frac{\partial j}{\partial \sigma}(0,0) \neq 0$. Thus we can use implicit function theorem! $\exists \sigma(\tau)$ for $|\tau|$ small so that $j(\tau, \sigma(\tau)) = 0$.

Now we can compute first variation:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} j(\tau, \sigma(\tau)) = 0$$

$$j_\tau(\tau, \sigma(\tau)) + j_\sigma(\tau, \sigma(\tau))\sigma'(\tau) = 0$$

Thus, $\sigma'(0) = \frac{-j_\tau(0,0)}{j_\sigma(0,0)}$

$$= \frac{-\int_\Omega g(u_0)v\,\mathrm{d}x}{\int_\Omega g(u_0)w\,\mathrm{d}x}$$

Here's the curve of competitors:

$$V(\tau) := u_0 + \tau v + \sigma(\tau)w \in \mathcal{A}\,\forall |\tau| \,\text{small}$$

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \frac{1}{2}\int_\Omega |\nabla V|^2\,\mathrm{d}x$$

$$0 = \left.\frac{\partial}{\partial \tau}\right|_{\tau=0} \frac{1}{2}\int_\Omega (\nabla u_0 + \tau\nabla v + \sigma(\tau)\nabla w, \nabla u_0 + \tau\nabla v + \sigma(\tau)\nabla w)\,\mathrm{d}x$$

$$0 = \int_\Omega \nabla u_0 \cdot \nabla v + \nabla u_0 \cdot \nabla w \sigma'(0)\,\mathrm{d}x$$

$$0 = \int_\Omega \nabla u_0 \nabla v\,\mathrm{d}x - \int_\Omega g(u_0)v\,\mathrm{d}x \frac{\int_\Omega \nabla u_0 \cdot \nabla w\,\mathrm{d}x}{\int_\Omega g(u_0)w\,\mathrm{d}x}$$

$$0 = \int_\Omega \nabla u_0 \cdot \nabla v - \lambda g(u_0)v\,\mathrm{d}x \overset{IBP}{\Longrightarrow} -\Delta u_0 = \lambda g(u_0)\,\text{in}\,\Omega$$

$-\Delta u_0 = \lambda g(u_0)$ in Ω is a nonlinear eigenvalue problem.