M623 Geometric Topology I

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Monday, 8/25/2025

Textbook: *Characteristic Classes* by Milnor and Stasheff. Hereafter referred by MS.

Read Chapter 1 and 2 of MS.

Definition (n-manifold)**.** Two different variants: embedded and abstract.

Abstract: $(M, \mathcal{A})$ where $\mathcal{A}$ is an atlas.

Embedded: $M \subset \mathbb{R}^A$. Here, $A =$ index set, $\mathbb{R}^A = \text{func}(A, \mathbb{R})$ with the product topology.

M Hausdorff space, $U \subset M$ open, $V \subset \mathbb{R}^n$ open.

Chart $\phi : U \xrightarrow{\approx} V$ homeomorphism.

Parameterization (ptz) $h : V \xrightarrow{\approx} U$

We want some calculus.

Let open $V \subset \mathbb{R}^n$.

A function $f : V \to \mathbb{R}$ is *smooth* if all partials of all orders exist: $\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_p}}$.

$f : V \to \mathbb{R}^A$ is *smooth* if f_α smooth $\forall \alpha \in A$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "f_\alpha"'] & \mathbb{R}^A \arrow[r, "pr_\alpha"] & \mathbb{R} \end{tikzcd}$$

We can go from abstract manifold to embedded manifold.

Let $A = C^\infty(M, R)$.

$M \xrightarrow{i} \mathbb{R}^A$ where $i(x) = (f \mapsto f(x))$.

We can go to the reverse direction easily once we have all the definitions.

Definition. Two charts $(\phi_1 : U_1 \to V_1)$ and $(\phi_2 : U_2 \to V_2)$ are compatible (or smoothly compatible) if $\phi_2 \circ \phi_1^{-1}$ is smooth. Explicitly,

$\phi_1(U_1 \cap U_2) \xrightarrow{\phi_2 \circ \phi_1^{-1}} \phi_2(U_1 \cap U_2)$ needs to be smooth.

Definition. Parameterization $h : V \to U$ is *smooth* (assume $M \subset \mathbb{R}^A$) if:

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[rrr, bend right, "h"'] & U \arrow[r, hook] & M \arrow[r, hook] & \mathbb{R}^n \end{tikzcd}$$

is smooth.

and has rank n. ie, $\forall v \in V$ the Jacobian:

$$dh = \left(\frac{\partial_\alpha h}{\partial x_j}(v) \right)$$

has rank n.

eg $x \mapsto x^3$ is a parameterization which is not smooth, since the Jacobian has rank 0 at 0.

Now we can properly define manifolds.

Definition (Embedded Smooth n-Manifold)**.** $M \subset \mathbb{R}^A$ so that $\forall x \in M$ there exists a smooth rank n parameterization $h : V \to U \ni x$.

We assume M is Hausdorff.

We can now define a Cateogry of Embedded Manifolds.

Definition (Category of Embedded Manifolds)**.** Embmfld.

Objects: embedded $M \subset \mathbb{R}^A$ of $\dim n$ for some n.

Morphisms: Smooth Maps (has to be defined carefully, restricting in Euclidean space).

Diffeomorphism = invertible morphism.

Let $(M \subset \mathbb{R}^A), (N \subset \mathbb{R}^B)$. $f : M \to N$ is smooth if *locally smooth*, meaning $\forall x \in M, \exists$ smooth parameterization $h : V \to U \ni x$ such that $V \to U \hookrightarrow M \xrightarrow{f} N \to \mathbb{R}^B$ is smooth.

Now we can define abstract manifold independend of embedded manifolds.

Definition (Abstract Manifold)**.** Let M be Hausdorff. An *n-atlas* on M is a set $\mathcal{A} = \left\{ \phi_\alpha : U_\alpha \xrightarrow{\approx} V_\alpha \subset \mathbb{R}^n \right\}$ of compactible n-charts such that $\{U_\alpha\}$ covers M.

Atlas $\mathcal{A}$ and $\mathcal{A}'$ are compatible if all charts are.

Fact: Every atlas is contained in a unique maximal atlas.

Then an abstract manifold is $(M, \mathcal{A})$ with a maximal n-atlas.

Wednesday, 8/27/2025

Recall: embedded n-manifold $M \subset \mathbb{R}^A$: $\forall x \in M, \exists$ smooth, rank n parameterization $h : V \to U \subset M$ such that $x \in U$. We assume M is Hausdorff.

Abstract n-manifold: $(M, \mathcal{A})$ where $\mathcal{A}$ is an n-atlas, so $\mathcal{A} = \{\text{charts } \phi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha\}$ such that $\{U_\alpha\}$ cover M and $\{\phi_\alpha\}$ smoothly compatible. We assume M is Hausdorff.

Remark. If we have an abstract manifold we have a surjective map $\coprod V_\alpha \overset{\coprod \phi_\alpha^{-1}}{\twoheadrightarrow} M$.

Then we can define $M \cong \frac{\coprod V_\alpha}{\sim}$. This gives us another definition of a manifold.

Exercise. Define smooth $f : (M, \mathcal{A}) \to (N, \mathcal{B})$.

Not hard, just annoying to get the definitions right!

Theorem 1. Categories of abstract manifolds and embedded manifolds are equivalent.

$$\text{EmbMflds} \simeq \text{absMflds}$$

Recall equivalent categories:

Definition. Categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent (Notation: $\mathcal{C} \simeq \mathcal{D}$): If there are functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{C}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to the respective identities.

We need some more definitons.

Definition. A skeleton of $\mathcal{C}$ is $\operatorname{Sk}\mathcal{C} \subset \mathcal{C}$ is a full subcategory $\forall c \in \mathcal{C}, \exists! c' \in \operatorname{Sk}\mathcal{C}$ such that $c \cong c'$.

$\mathcal{A} \subset \mathcal{B}$ is full if $\forall a, a' \in \operatorname{Ob}\mathcal{A}, \mathcal{A}(a, a') \xrightarrow{\cong} \mathcal{B}(a, a')$

For example, let $\mathcal{C}$ = finite sets. Then $\operatorname{Sk}\mathcal{C} = \{1\}, \{1, 2\}, \{1, 2, 3\}, \cdots$

Theorem 2. $\mathcal{C} \simeq \mathcal{D} \iff \operatorname{Sk}\mathcal{C} \cong \operatorname{Sk}\mathcal{D}$.

Note that $\mathcal{C} \simeq \operatorname{Sk}\mathcal{C}$ so one direction is trivial.

Lemma 3 (1.1)**.** Let h and h' be smooth rank n on $M \subset \mathbb{R}^A$. Then $h^{-1} \circ h'$ is smooth (thus a diffeomorphism).

Let V and V' be the domain of h and h' respectively. Then $h^{-1} \circ h' : (h')^{-1}(V \cap V') \to h^{-1}(V \cap V')$

Corollary 4. $\mathcal{A} = \{h^{-1} \mid h \text{ parameterization}\}$ is n-atlas on M.

This gives us EmbMflds $\to$ AbstMflds.

Proof. This is the proof of lemma 1.1, lemma 3 in the notes.

Assume $V = V'$. WTS: $(h')^{-1}V \to h^{-1}(V)$ is smooth.

For $x \in V$ choose $\alpha_1, \cdots, \alpha_n \in A$ such that $\det\left(\frac{\partial \alpha_i}{\partial x_j}(x)\right) \not\equiv 0$.

We have:

$$\begin{tikzcd}
M \arrow[r, hook] & \mathbb{R}^A \arrow[d, "pr_{\alpha_1 \cdots \alpha_n}"] \\
V \arrow[u, "h"] \arrow[r, dotted] & \text{subset of } \mathbb{R}^n
\end{tikzcd}$$

Then, by the inverse function theorem, the dotted map is locally invertible.

$h^{-1} \circ h' = (pr \circ inc \circ h)^{-1} \circ inc \circ pr \circ h'$ near $h^{-1}x$.

□

Given abstract $(M, \mathcal{A})$, let $A = C^\infty(M, \mathbb{R})$ smooth functions.

$i : M \to \mathbb{R}^A, x \mapsto (f \to f(x))$.

Let $M_1 = i(M)$.

Lemma 5 (1.5)**.** $M_1 \subset \mathbb{R}^A$ is EmbMfld. $M \xrightarrow{i} M_1$ is diffeomorphism.

Definition of tangent vector, tangent space and tangent bundle

Definition (Tangent Vector)**.** is velocity vector of a curve.

We have defined morphisms. Consider the embedded case: suppose we have smooth $\gamma : \mathbb{R} \to M \subset \mathbb{R}^A$. Then,

$$\gamma'(0) = \lim_{h \to 0} \frac{\gamma(h) - \gamma(0)}{h} \in \mathbb{R}^A$$

is a tangent vector

Definition (Tangent Space)**.** Suppose $x \in M \subset \mathbb{R}^A$, an n-dim embedded manifold. T_xM = tangent space of M at x. This is:

$$\{\gamma'(0) \mid \gamma(0) = x\} \subset \mathbb{R}^A$$

an n-dim subspace.

We are going to bundle this together.

Definition (Tangent Bundle)**.** $TM = \big\{(x, v) \in M \times \mathbb{R}^A \mid v \in T_xM\big\}$.

By definition, $TM \subset M \times \mathbb{R}^A$ so this is in fact a topological space.

We have a projection map $TM \xrightarrow{\pi} M$ by $(x, v) \mapsto x$.

Remark. Fibers of π, $\pi^{-1}(x)$ are vector spaces: $\pi^{-1}(x) = T_xM$.

Then, $TM = \bigcup_{x \in M} \{x\} \times T_xM$.

Abuse of notation lets us write this as $\bigcup T_xM$.

Thus, tangent bundle is in fact a bundle of tangents.

What about abstract manifolds $(M, \mathcal{A})$?

We can define TM as follows:

- $M \subset \mathbb{R}^{C^\infty(M,\mathbb{R})}$.
- $TM = \frac{\coprod V_\alpha \times \mathbb{R}^n}{\sim}$

- T_xM = velocity vector of curves.
- derivations.

Suppose we have smooth function between manifolds $f : M \to N$. $\forall x \in M$ we can define linear $\mathrm{d}f_x : T_xM \to T_{f(x)}N$, $\gamma'(0) \mapsto (f \circ \gamma)'(0)$. $\mathrm{d}f_x$ is a map between vector spaces, so it is a linear transformation. It is the 'Jacobian'.

Then we have $\mathrm{d}f : TM \to TN$ such that $\mathrm{d}f(x, v) = \mathrm{d}f_x(v)$.

We also have the chain rule: $\mathrm{d}(f \circ g) = \mathrm{d}f \circ \mathrm{d}g$

Friday, 8/29/2025

No class next week!

Manifold constructed by:

- open subset of $\mathbb{R}^n$
- Subset double torus $\subset \mathbb{R}^3$
- Quotients: $P^n = \mathbb{R}P^n = S^n/x \sim -x$
- Lie groups/ matrix group, eg closed subgroups of $\mathrm{GL}_n \mathbb{R} \underset{\text{open}}{\subset} M_n\mathbb{R} = \mathbb{R}^{n^2}$
- Zero sets.
  - regular values
  - transversality
  - smooth varieties

Definition. $t_0 \in \mathbb{R}$ is a regular value of $f : M \to \mathbb{R}$ if $\forall x \in f^{-1}t_0, \mathrm{d}f_x$ is onto.

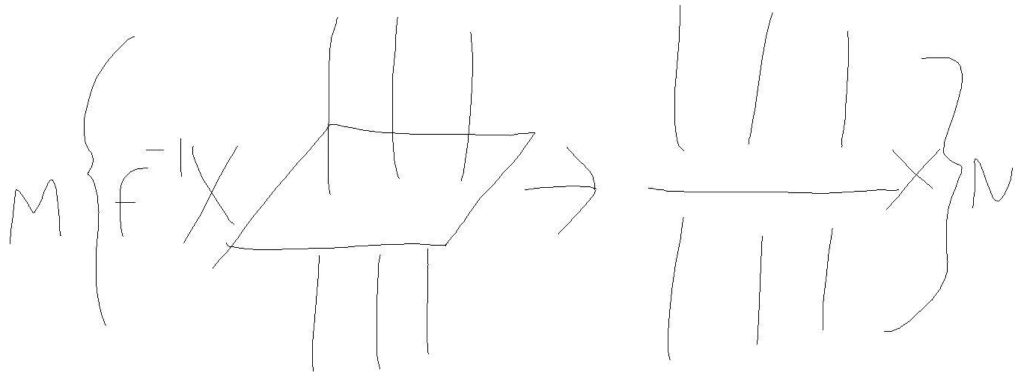
f^{-1}(regular value) is a submanifold of M.

Consider $S^n \subset \mathbb{R}^{n+1}$, and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $x \mapsto x_1^2 + \cdots + x_{n+1}^2$.

1 is a regular value $f^{-1}1 = S^n$.

Definition. Let $f : M \to N \supset X$ submanifold.

$f \pitchfork X$, f is *transverse* to X if $\forall m \in f^{-1}X, T_{f(m)}N = T_{f(m)}X + \mathrm{d}f_m(T_mM)$.



Theorem 6. $f^{-1}X$ is a submmanifold of M.

Furthermore, $\dim N - \dim X = \dim M - \dim f^{-1}X$.

In fact, $\nu(f^{-1}X \hookrightarrow M) \to \nu(X \hookrightarrow N)$ as vector space isomorphism on fibers.

[insert picture later]

Now, suppose F is a topological space.

Definition. A fiber bundle with fiber F:

Let $E \xrightarrow{\pi} B$ be a continuous map suuch that $\forall b \in B, \exists$ open $b \in U \subset B$ and:

$$\begin{tikzcd} U \times F \arrow[rr, "h", "\approx"'] \arrow[dr, "pr_U"'] & & \pi^{-1}U \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

h fiber preserving homeomorphism. $\forall b' \in U$, $F \cong F \times b' \xrightarrow{\approx} F_{b'} := \pi^{-1}(b')$.

Write: $\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$

$$\begin{tikzcd} I \arrow[r] & Mob \arrow[d] \\ & S^1 \end{tikzcd}$$

eg $B \times F \to B$ trivial bundle.

Chapter 2 of MS

Definition. A real vector bundle ξ over B is:

$$\xi = \left(\begin{array}{c} E \\ \downarrow \pi \\ B \end{array}, \forall b \in B, \pi^{-1}b = F_b \text{ is a fin. dim vector space.} \right)$$

$F_b \times F_b \to F, \mathbb{R} \times F_b \to F$ satisfies 8 axioms s.t.

$\forall b \in B, \exists b \in U \subset B$ and $n \geq 0$ and $\begin{tikzcd} U \times \mathbb{R}^n \arrow[rr, "h", "\approx"'] \arrow[dr] & & \pi^{-1}U \arrow[dl] \\ & U & \end{tikzcd}$.

$\mathbb{R}^n \cong b \times \mathbb{R}^n \xrightarrow[\approx]{h} \pi^{-1}b$ is an isomorphism of vector spaces.

If B is connected then n is constant.

'rank n vector bundle'.

n-plane bundle.

Another thing MS does is write this: $\xi = \begin{array}{c} E(\xi) \\ \big\downarrow{\scriptstyle \pi(\xi)} \\ B(\xi) \end{array}$ for vector bundle which is very precise.

Isomorphism of vector bundles over B

.

Consider two bundles ξ and η and we have the homeomorphism

$$\begin{tikzcd} E(\xi) \arrow[rr, "\approx"] \arrow[dr] & & E(\eta) \arrow[dl] \\ & B & \end{tikzcd}$$

vector space isomorphism on the fibers.

Examples of vector bundles

We have the trivial bundle $\underline{\mathbb{R}^n} = \underline{\mathbb{R}}^n_B = \varepsilon^n_B = \begin{array}{c} B \times \mathbb{R}^n \\ \big\downarrow \\ B \end{array}$

We have tangent bundles:

$$\tau_M = \left\{ \begin{array}{c} TM \\ \big\downarrow{\scriptstyle \pi} \\ M \end{array}, T_x M \right\}$$

Definition. M is parallelizable if τ_M is trivial.

S^1 is paralellizable.

Lie groups are parallelizable eg S^3.

S^2, or S^{2n} in general not parallelizable via the hairy ball theorem.

We also have normal bundles. Consider $M \subset \mathbb{R}^N$.

$\nu(M \subset \mathbb{R}^n) = \{(x, v) \in M \times \mathbb{R}^n \mid x \in M, v \in (T_x M)^\perp\}$

$\nu(S^2 \hookrightarrow S^3) \leftarrow S^2 \times \mathbb{R}$ is trivial, the map is $(x, tx) \leftarrow\!\shortmid (x, t)$.

Tautological bundle over P^n: $\gamma_n^1 =$ $\begin{tikzcd} \mathbb{R} \arrow[r] & E(\gamma_n^1) \arrow[d] \\ & P^n \end{tikzcd}$

Note that $P^n = S^n / x \sim -x =$ lines through O in $\mathbb{R}^{n+1}$.

$E(\gamma_n^1) = \big\{(\{x, -x\}, v) \in P^n \times \mathbb{R}^{n+1} \mid v \in \mathbb{R}x\big\}$.

$E(\gamma_n^1) \xrightarrow{\pi} P^n, (\{x, -x\} \mapsto \{x, -x\})$. Essentially, point on line $\mapsto$ line.

This tautological bundle is non-trivial.

Monday, 9/8/2025

Last week was a break.

HWK: an exercise from ch2. (C, D, E are recommended).

Recall: a vector bundle ξ is $\begin{tikzcd} \mathbb{R} \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$ meaning fibers of π are k-dimensional vector spaces.

Definition. A section of ξ is actually a section of π.

$s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$.

Section looks like this:

$$\begin{tikzcd} \mathbb{R}^k \arrow[r] & E \arrow[d, "\pi"] \\ & B \arrow[u, dotted, bend left, "s"] \end{tikzcd}$$

Section of TM =: vector field.

There's also the zero section $z : B \to E$ given by $b \mapsto 0 \in \pi^{-1} b$.

$\begin{tikzcd} E \arrow[d, "\pi"] \\ B \arrow[u, dotted, bend left, "z"] \end{tikzcd}$ homotopy inverses.

Now we show there is some twisting.

$\begin{tikzcd} E_0 \arrow[d] \\ B \end{tikzcd}$ $= E - z(B)$. B trivial implies $E_0 \cong B \times (\mathbb{R}^k \setminus e) \simeq B \times S^{k-1}$.

We have the tautological line bundle:

$$\begin{array}{ccc} R \longrightarrow E & = \{([x], v) \mid v \in \mathbb{R}x\} & \subset P^n \times \mathbb{R}^{n+1} \\ \downarrow & & \\ P^n & = S^n / x \sim -x & \end{array}$$

We can think of it like $(\text{line}, \text{point on line}) \in E$.

For example, consider P^1. This gives us the open mobius strip.

Theorem 7 (2.1)**.** γ_n^1 is nontrivial for $n \geq 1$.

Proof. $E(\gamma_n^1)_0$ is connected $\iff \not\cong P^n \times S^0$. □

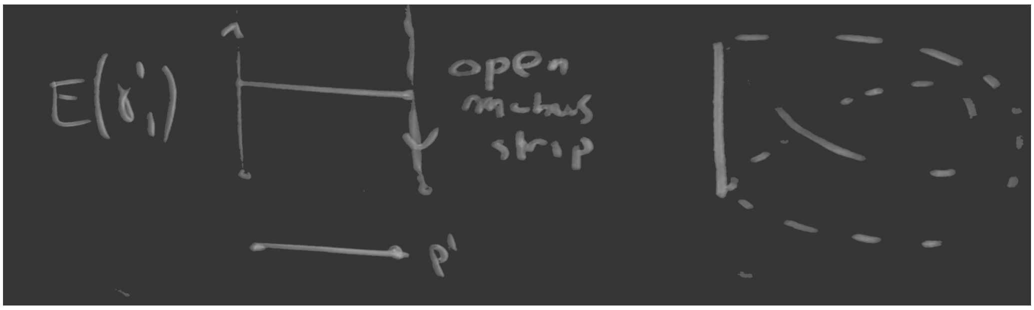


Figure 1:

Definition. A metric on a vector bundle ξ is $g : E \times_B E \to \mathbb{R}$ such that $\forall b \in B, \pi^{-1}b \times \pi^{-1}b \to \mathbb{R}$ is an inner product.

Recall: pullback of $\begin{array}{ccc} & & B \\ & & \downarrow \beta \\ A & \xrightarrow{\alpha} & C \end{array}$ is $A \times_C B = \{(a, b) \mid \alpha(a) = \beta(b)\} \subset A \times B$.

Also see: a vector bundle $E \to B$ needs all fibers to be vector spaces. For a metric we want them to be inner product spaces.

A bundle with metric is often callled a Euclidean vector bundle.

Examples: A *Riemannian manifold* is TM with a smooth metric [g is smooth].

If $M^n \subset \mathbb{R}^N$ we can use the inner product inherited from $\mathbb{R}^N$ so it is a riemannian manifold.

eg the trivial bundle has a metric: $(B \times \mathbb{R}^n) \times_B (B \times \mathbb{R}^n) \to \mathbb{R}^n$ which looks like $((b, v), (b, w)) \mapsto v \cdot w$.

If $M^n \subset \mathbb{R}^N$, $TM = \left\{(x, v) \in M \times \mathbb{R}^N \mid v = \gamma'(0), \gamma(0) = x\right\}$

$\|(x, v)\| = \|v\|, g((x, v), (x, w)) = v \cdot w$.

Then $\|\cdot\| : E \to \mathbb{R}_{\geq 0}$ given by $\|v\| := \sqrt{g(v, v)}$.

Theorem 8 (Exercises, ch2)**.** Suppose B is paracompact. We can look at Isomorphism classes of Euclidean vector bundles over B, forget the metric to get isomorphism classes of vector bundles over B:

$$\left\{\begin{array}{c}\text{iso class of euclidean} \\ \text{vector bundle over } B\end{array}\right\} \xrightarrow{\text{forget } g} \left\{\begin{array}{c}\text{iso class of} \\ \text{vector bundle over } B\end{array}\right\}$$

This is an isomorphism.

Definition. Sections $s_1, \cdots, s_n$ of rank n vector bundle given by $\begin{array}{c} E \\ \downarrow \uparrow s_i \\ B \end{array}$ are linearly independent (l.i) if $\forall b \in B, \{s_1(b), \cdots, s_n(b)\}$ is linearly independent in $\pi^{-1}(b)$.

Theorem 9 (2.2)**.** rank n vector bundle ξ is trivial iff ξ has n l.i. sections.

Proof. $\implies$: $s_i(b) := (b, \underline{e_i}) \in B \times \mathbb{R}^n$.

$\impliedby$: define $f : B \times \mathbb{R}^n \to E$ by $(b, \sum a_i e_i) \mapsto \sum a_i s_i(b)$ □

eg T^2 has 2 l.i. sections, thus $TT^2 \cong T^2 \times \mathbb{R}^2$.

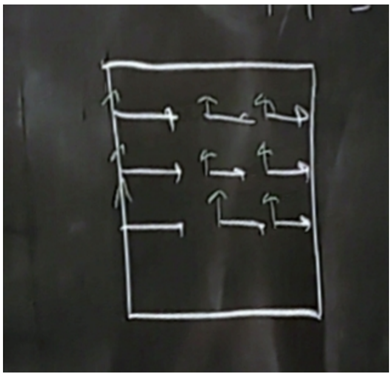


Figure 2:

Wednesday, 9/10/2025

Chapter 3: New bundles

Homeowrk: pick up problems from chapter 3 (and chapter 2).

Abstract definition of bundle (Steenrod, see D-Kirk 5.2).

Let G be a topological group, F a space, $G \curvearrowright F$

Topological group meaning: G topological group means G is a group and a space such that $G \times G \to G, (a, b) \mapsto ab$ and $G \to G, a \mapsto a^{-1}$ are continuous.

Action of G on F: $G \times F \to F$ given by $ef = f$ and $(gg')f = g(g'f)$.

Definition. A fiber bundle with structure group G and fiber F [(G, F)-bundle] is a map with:

Map $\begin{array}{c} E \\ \downarrow \\ F \end{array}$

Atlas $\mathcal{A} = \{\phi : U_\phi \times F \xrightarrow{\approx} \pi^{-1}U_\phi\}$

Transition functions $\Theta = \{\theta_{\phi,\psi} : U_\phi \cap U_\psi \to G \mid \phi, \psi \in \mathcal{A}\}$

such that:

1) $\{U_\phi\}$ open cover of B.
2) Fiber preserving homeomorphism:

   the following diagram commutes:
$$\begin{tikzcd} U_\phi \times F \arrow[rr, "\approx"] \arrow[dr] & & \pi^{-1}U_\phi \arrow[dl] \\ & U_\phi & \end{tikzcd}$$
3) $b \in U_\phi \cap U_\psi, f \in F \implies \psi(b,f) = \phi(b, \theta_{\phi,\psi}(b)f)$
4) $\theta_{\phi,\psi}(b) = \theta_{\phi,\chi}(b)\theta_{\chi,\psi}(b)$

Examples:

G trivial group implies the bundle is a trivial bundle, $\begin{array}{c} B \times F \\ \downarrow \\ B \end{array}$

$G = \mathrm{GL}(n,\mathbb{R}), F = \mathbb{R}^n$ gives us the rank n vector bundle. Let $b \in B$, choose $\phi, b \in U_\phi$. Use the atlas to find bijection $\pi^{-1}b \cong \mathbb{R}^n$. This gives us a vector space on $\pi^{-1}b$ independent of the choice of U_ϕ by the 3rd condition.

If the G-action on F is *effective*, meaning every non-trivial action does something, meaning there is $f \in F$ such that $gf \neq f$ for every $g \in G \setminus \{e\}$, then we don't need condition 4.

If $G = \mathrm{O}(n)$ and $F = \mathbb{R}^n$ then we have a vector bundle with a metric.

If $G = \mathrm{GL}(n,\mathbb{R})^+$ and F is $\mathbb{R}^n$ then we have an oriented vector bundle.

If $G = S_F = \mathrm{Aut}(F)$ where F is discrete, then we have a cover.

For discrete G with $F = G$ then we have a regular G-cover.

If $G = \mathrm{Spin}(n), F = \mathbb{R}^n$ then we have a vector bundle with spin structure.

Now we start chapter 3. We can do a lot of things on vector spaces, like tensor products. This lets us do stuff with vector bundles as well.

Some basic constructions involving vector bundles:

1) Restriction: Let ξ be a vector bundle, $\overline{b} \hookrightarrow B$. Then we can let $\xi\big|_{\overline{B}} = \begin{array}{c} \pi^{-1}\overline{B} \\ \downarrow \\ \overline{B} \end{array}$

$$\begin{tikzcd} & \xi \arrow[d, equal] \\ & E \arrow[d, "\pi"] \\ \overline{B} \arrow[r, hook] & B \end{tikzcd}$$

2) Induced bundles (= Pullback bundle) Let ξ be a vector bundle, and $B_1 \xrightarrow{f} B$. We can *pullback* the bundle and get $f^*\xi$:

$$\begin{tikzcd} f^*E = B_1 \times_B E \arrow[r] \arrow[d] & E \arrow[d] \\ B_1 \arrow[r, "f"] & B \end{tikzcd}$$

in fact $\xi\big|_{\overline{B}} = \mathrm{inc}^*\,\xi$.

Definition. Bundle map $g : \eta \to \xi$ [both n-plane] is given by a commutative diagram which is isomorphism on fibers:

$$\begin{tikzcd} E(\eta) \arrow[r, "g"] \arrow[d] & E(\xi) \arrow[d] \\ B(\eta) \arrow[r, "\overline{g}"] & B(\xi) \end{tikzcd}$$

Lemma 10 (3.1)**.** $\eta \cong \overline{g}^*$ as vector bundle over $B(\eta)$.

$$\begin{tikzcd} E(\eta) \arrow[rr, "\approx"] \arrow[dr] & & \overline{g}^*E(\xi) \arrow[dl] \\ & B(\eta) & \end{tikzcd}$$

Proof. We just need to define the map.

$$E(\eta) \to B(\eta) \times_{B(\xi)} E(\xi)$$

$$e \mapsto (\pi(e), g(e))$$

$\square$

pullback stuff works for (G, F)-bundles.

Friday, 9/12/2025

Today we finish chapter 3.

We can study construction of new vector bundles in the following ways:

a) *Restriction*: $\xi\big|_{\overline{B}}$ for $\overline{B} \subset B \leftarrow E$

b) *Pullback*: $f^*\xi$ for $\overline{B} \xrightarrow{f} B \leftarrow E$

c) *Product*: $\xi_1 \times \xi_2$.

$$\begin{tikzcd} F_b(\xi_1) \times F_b(\xi_2) \arrow[r] & E(\xi_1) \times E(\xi_2) \arrow[d] \\ & B(\xi_1) \times B(\xi_2) \end{tikzcd}$$

eg $T(M_1 \times M_2) = TM_1 \times TM_2$.

d) *Whitney Sum*: We keep the base space the same. Let ξ_1, ξ_2 be vector bundles over the same base space B. Then we can define the whitney sum as the pullback of the diagonal map to the product:

$$\xi_1 \oplus \xi_2 := \Delta^*(\xi_1 \times \xi_2)$$

$B \xrightarrow{\Delta} B \times B$ is $b \mapsto (b, b)$.

For example, in $S^2 \hookrightarrow \mathbb{R}^3$, the whitney sum of the tangent bundle and the normal bundle gives us the trivial bundle: $\varepsilon^3_{S^2} = TS^2 \oplus \nu(S^2 \hookrightarrow \mathbb{R}^3)$.

e) *Subbundles, Quotients and Orthogonal Complements*: A subbundle η of ξ is $E(\eta) \subset E(\xi)$ such that $\pi\big|_{E(\eta)}$ is a vector bundle.

$$\begin{tikzcd}
F_b(\eta) \arrow[rr, hook] \arrow[d] & & F_b(\xi) \arrow[d] \\
E(\eta) \arrow[rr, hook] \arrow[dr] & & E(\xi) \arrow[dl] \\
& B &
\end{tikzcd}$$

In order to study quotient, we need *bundle morphisms.* We want the following diagram to be commutative and also want the map to be linear on fibers:

$$\begin{tikzcd}
E(\eta) \arrow[r] \arrow[d] & E(\xi) \arrow[d] \\
B(\eta) \arrow[r] & B(\xi)
\end{tikzcd}$$

Bundle morphism over B is different: we want the following commutative diagram to be linear on fibers:

$$\begin{tikzcd}
E(\eta) \arrow[rr] \arrow[dr] & & E(\xi) \arrow[dl] \\
& B &
\end{tikzcd}$$

An example: suppose we have smooth $f : M \to N$. Then we have bundle morphism:

$$\begin{tikzcd}
M \arrow[r, "df"] \arrow[d] & TN \arrow[d] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

and the bundle morphism $/M$:

$$\begin{tikzcd}
TM \arrow[rr, "\cong"] \arrow[dr] & & f^*TN \arrow[dl] \\
& M &
\end{tikzcd}$$

We can define quotient bundles from subbundles: subbundle η of ξ there exists quotient bundle ξ/η so that $F_b(\xi/\eta)$ are $F_b(\xi)/F_b(\eta)$. We have bundle map over B $\xi \to \xi/\eta$

Bundles $/B$ fform abelian category. We have the SES:

$$0 \to \eta \to \xi \to \xi/\eta \to 0$$

We now define normal bundles. Normal bundle of submanifold M of N is given by $\nu(M \hookrightarrow N) = \frac{(TN|_M)}{TM}$.

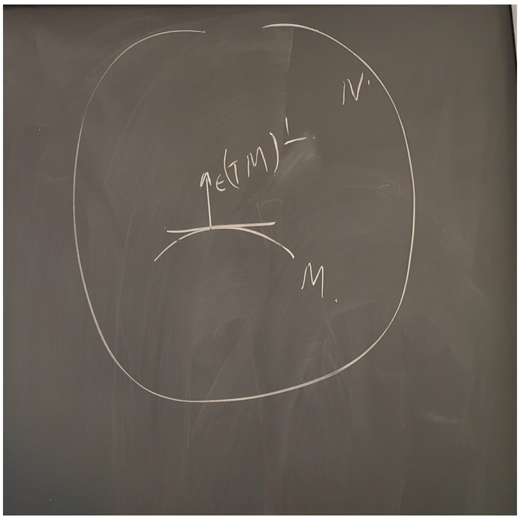


Figure 3:

If $N \subset \mathbb{R}^k$ (or N Riemannian metric space) then $(TM)^{\perp} \subset TN\big|_M$.

$$\begin{tikzcd} (TM)^{\perp} \arrow[r] \arrow[rr, bend right, "\cong"] & TN\big|_M \arrow[r] & \nu(M \hookrightarrow N) \end{tikzcd}$$

We have $(TN)_M = TM \oplus (TM)^{\perp}$.

If ξ is a bundle with metric and η is a subbundle then $\xi = \eta \oplus \eta^{\perp}$ and $\eta^{\perp} \cong \xi/\eta$.

If B is paracompact [eg $B \subset W$] then bundles over B form an exact category [meaning all SES split].

Reason: consider the following SES:

$$0 \to \alpha \to \beta \to \gamma \to 0$$

Since B is paracompact we can give β a metric. $\alpha^{\perp} \xrightarrow{\approx} \gamma$ so it splits.

This tells us: if $M \subset N$ and N has a Riemannian metric, then,

$TN\big|_M = TM \oplus TM^{\perp} \cong TM \oplus \nu(M \hookrightarrow N)$.

Definition. Smooth $f : M \to N$ is a immersion/submersion if $\forall x \in M$, df_x is injective/surjective.

For example, consider $S^1 \to \mathbb{R}^2$ given by $\bigcirc \to \infty$ is an immersion, since it's locally an embedding.

$TS^2 \to S^2$ is a submersion.

Let $f : M \to N$ be an immersion. Then, $\nu(f) = \frac{f^*TN}{TM}$.

If N has a metric then $TM \cong TN\big|_M \oplus \nu(f)$.

Tuesday, 9/16/2025

UCT, Cup and Cap Prodcuts

Let M be an abelian group. Then we have homology $H_i(X,A;M)$ and cohomology $H^i(X,A;N)$ abelian groups.

The cohomology $H^i(X,A;N)$ is the cohomology of the following cochain complex: $H^i(\mathrm{Hom}(S_\bullet(X,A),N))$

'Cohomology eats homology' via the following *Kronecker Pairing*:

$$\langle , \rangle : H^i(X,A;N) \otimes H_i(X,A;M) \to N \otimes_{\mathbb{Z}} M$$

$$[\phi] \otimes \left[\sum_i k_i \sigma_i \otimes m_i\right] \mapsto \sum_i k_i \varphi(\sigma_i) \otimes m_i$$

Now we do UCT. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}$-module, i.e. abelian group.

If $X = \mathbb{R}P^n$ then the cellular chain complex of $\mathbb{R}P^n$ is:

$$C_\bullet X = \mathbb{Z} \xrightarrow[0\ n \text{ odd}]{2\ n \text{ even}} \cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Thus, if n odd, then $H_i \mathbb{R}P^n = \begin{cases} \mathbb{Z}, & \text{if } i = 0, n; \\ \mathbb{Z}_2, & \text{if } i \text{ odd, } 0 < i < n; \\ 0, & \text{otherwise.} \end{cases}$

If coefficients are in $\mathbb{Z}_2$ then,

$$C_\bullet X \otimes \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}_2$$

Thus $H_i(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2$ for $0 \leq i \leq n$.

UCT states that the following is a split short exact sequence:

$$0 \to H_i X \otimes M \to H_i(X;M) \to \mathrm{Tor}(H_{i-1}X, M) \to 0$$

We can say three things about Tor:

Tor is a functor, $\mathrm{Tor} : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$.

If M, N are f.g. then $\mathrm{Tor}(M,N) \cong (\text{torsion } M) \otimes_{\mathbb{Z}} (\text{torsion } N)$

Definition. Find an exact sequence of free groups as follows:

$$0 \to F_1 \to F_0 \to M \to 0$$

Then $\mathrm{Tor}(M,N) = H_1(F_1 \otimes N \to F_0 \otimes N)$.

For example, $\mathrm{Tor}(\mathbb{Z}_2, \mathbb{Z}_2)$, we have following free groups:

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \to 0$$

Tensoring with $\mathbb{Z}_2$ to get the following: $\mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2$. Then H_1 is the kernel.

So, $\mathrm{Tor}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$.

Now we go back to geometry.

Suppose we have space X such that $H_{i-1}X = \mathbb{Z}_2 \oplus ?$

This gives us $H_i(X) \to \mathbb{Z}_2 \subset H_i(X; \mathbb{Z}_2)$.

Geometrically, consider $H_i(X; \mathbb{Z}_2) \to \mathrm{Tor}(H_{i-1}(X); \mathbb{Z}_2)$.

If there is $[a] \in \mathrm{Tor}(H_{i-1}X; \mathbb{Z}_2)$ with $2a = \partial b$ then section given by $[b] \leftarrow\!\shortmid [a]$

UCT works even if we change $\mathbb{Z}$ with a PID. For any PID R we can talk about R-modules M, then $H_i(X; M) \cong H_i(X; R) \otimes M \oplus \mathrm{Tor}^R(H_{i-1}(X; R), M)$.

We want the analogue of UCT for cohomology. This gives us the split exact sequence:

$$0 \to \mathrm{Ext}(H_{i-1}X, M) \to H^i(X; M) \to \mathrm{Hom}(H_iX, M) \to 0$$

Again, for n odd consider the chain complex:

$$C_\bullet \mathbb{R}P^n = \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \cdots \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

For cochain complex we'd simply reverse the arrows:

$$C^\bullet \mathbb{R}P^n = \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow \cdots \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

$H_i \mathbb{R}P^n = \mathbb{Z}$ for $i = 0, n$ and $\mathbb{Z}_2$ for $0 < i < n, n$ odd.

$H^i(\mathbb{R}P^n; \mathbb{Z}) = \mathbb{Z}$ for $i = 0, n$ and $\mathbb{Z}_2$ for $0 < i < n, n$ even.

We have: $\mathrm{Ext}(\text{Free}, M) = 0$.

In general, $\mathrm{Ext}(A, B)$ is given by: resolve A, apply $\mathrm{Hom}(-, B)$ cohomolgy.

Suppose $0 \to F_1 \to F_0 \to A \to 0$.

Then, $\mathrm{Hom}(F_1, B) \xleftarrow{\partial^1} \mathrm{Hom}(F_0, B)$.

Thus $\mathrm{Ext}(A, B) = \operatorname{coker} \partial^1$.

If A, B are finitely generated then $\mathrm{Ext}(A, B) \cong (\text{torsion } A) \otimes B$.

Now, suppose R is a commutative ring.

Then $H^i(X; R) = H^i(\mathrm{Hom}_{\mathbb{Z}}(X_\bullet X, R))$

But might be more in the spirit of how we are doing this to do the following:

$H^i(X;R) = H^i(\mathrm{Hom}_R(S_\bullet(X;R),R))$

For R-modules M,

$H^i(X;M) = H^i(\mathrm{Hom}_{\mathbb{Z}}(S_\bullet X, M)) = H^i(\mathrm{Hom}_R(S_\bullet(X;R),M))$

Then, $H^*(X;R)$ is a graded commutative ring under the cup product.

$H^*(X;R)$ is a graded commutative ring meaning we can write:

$H^*(X;R) = \bigoplus_{i\geq 0} H^i(X;R)$ and we have $H^i(X;R)\otimes_R H^j(X;R) \to H^{i+j}(X;R)$

Commutative graded ring meaning $\alpha\cup\beta = (-1)^{|\alpha||\beta|}\beta\cup\alpha$.

For De Rham cohomology,

$H^i_{\mathrm{DR}}(M;\mathbb{R})\otimes H^j_{\mathrm{DR}}(M;\mathbb{R})$ we have $\alpha\otimes\beta\mapsto[\alpha\wedge\beta]$

We also have: $H_*(M;R)$ is a graded module over $H^*(M;R)$ w.r.t. cap product.

For $\alpha\in H^i(M;R)$ and $z\in H_j(M;R)$ then $\alpha\cap z\in H_{j-i}(M;R)$.

So, cap product by α eats i dimensions from z.

We also have $\langle\alpha\cup\beta, z\rangle = \langle\alpha,\beta\cap z\rangle$.

If $f: X\to Y$ is continuous, we have a ring map $f^*: H^*(Y;R)\to H^*(X;R)$ by $f^*(\alpha\cup\beta) = f^*\alpha\cup f^*\beta$.

Poincaré Duality: if M^n is closed and oriented and connected then $H_nM\cong\mathbb{Z}$. Choose generator $[M]\in H_nM$.

Then we have isomorphism $\cap[M]: H^iM\xrightarrow{\cong}H_{n-i}M$

Another fact:

$$\frac{H^iM}{\text{torsion}}\otimes\frac{H^{n-i}M}{\text{torsion}}\to\mathbb{Z}$$

is a nonsingular perfect pairing: $\alpha\otimes\beta$ is given by $(\alpha\cup\beta)[M]\in\mathbb{Z}$.

Recall $A\times B\to\mathbb{Z}$ is perfect $\iff$ $A\xrightarrow{\cong}\mathrm{Hom}(B,\mathbb{Z})$ and $B\xrightarrow{\cong}\mathrm{Hom}(A,\mathbb{Z})$ are isomorphism.

In $\mathbb{C}P^n = e^0\cup e^2\cup\cdots\cup e^{2n}$ we have $H^*\mathbb{C}P^n\cong\mathbb{Z}[\alpha]/\alpha^{n+1}$, with $\deg\alpha = 2$.

This is a truncated polynomial ring.

We can prove this by Poincaré duality and induction on n.

We also have Kunneth Theorem. If R is a field, then:

$$H^*(X;R)\otimes H^*(Y;R)\xrightarrow{\cong}H^*(X\times Y;R)$$

It is only an injection for general ring.

Wednesday, 9/17/2025

HWK due 9/29.

4 Exercises: 1 from Ch2, 1 from Ch3, 2 from Ch4.

Today we finish chapter 3, construction of bundles.

We skipped part f on Friday.

Vector Spaces	Vector Bundle
$V \otimes W$	$\xi \otimes \eta$
$\mathrm{Hom}(V, W)$	$\mathrm{Hom}(\xi, \eta)$
$V^* = \mathrm{Hom}(V, \mathbb{R})$	$\xi^* = \mathrm{Hom}(\xi, \epsilon^1_B)$
$\Lambda^k V$	$\Lambda^k \xi$
$\Lambda^* V$	$\Lambda^* \xi$

Table 1: Anythong we can do on Vector Spaces, we can do in Vector Bundles.

As for $\mathrm{Hom}(\xi, \eta)$ we assume base space is the same:

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l) \arrow[r] & E\,\mathrm{Hom}(\overset{k}{\xi}, \overset{l}{\eta}) \arrow[d] \\ & B \end{tikzcd}$$

Here $E\,\mathrm{Hom}(\xi, \eta) =$ [roughly] $\bigcup_{b \in B} \mathrm{Hom}_{\mathbb{R}}(F_b(\xi), F_b(\eta))$

$:= \coprod_{\text{open } U \subset B,\, \xi|_U,\, \eta|_U \text{ trivial}} U \times \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l) / \sim.$

Cotangent Bundle

Let M^n be a smooth n-manifold.

Definition (Cotangent Bundle)**.** Is dual to the tangent bundle: $T^*M := (TM)^*$.

We can take exterior power to get differential k forms:

$$\begin{tikzcd} \Lambda^k \mathbb{R}^n \arrow[r] & \Lambda^k T^*M \arrow[d] \\ & M \arrow[u, bend right, dotted] \end{tikzcd}$$

Differential k-form on $M, \omega \in \Gamma(\Lambda^k T^*M)$ smooth section.

$$\begin{array}{lll} \Lambda^* \mathbb{R}^n \to & \Lambda^* T^*M & \\ & \downarrow & \leftarrow \text{wedge product.} \\ & M & \end{array}$$

In fact, $\Gamma(\Lambda^* T^*M)$ is a graded algebra, $\Omega^* M$.

Chapter 4

Now we start on Characteristic Classes.

Definition (Stiefel-Whitney Classes)**.** have these 4 axioms:

1) $\forall$ vector bundle ξ, assign $\mathrm{w}_i(\xi) \in H^i(B(\xi); \mathbb{F}_2)$ so that $\mathrm{w}_0(\xi) = 1$ and $\mathrm{w}_i(\xi) = 0$ for $i > n$ when ξ is a an n-plane bundle.
2) *Naturality*: For continuous $f : B' \to B(\xi)$, we have $\mathrm{w}_i(f^*\xi) = f^*(\mathrm{w}_i\, \xi) \in H^i(B; \mathbb{F}_2)$. [First one is the pullback on the bundle, second one is the induced map on the cohomology.]
3) *Whitney Sum Formula*: If ξ, η are vector bundles over B we have: $\mathrm{w}_k(\xi \oplus \eta) = \sum_{i+j=k} \mathrm{w}_i(\xi) \cup \mathrm{w}_j(\eta)$.
4) $0 \neq \mathrm{w}_1(\gamma_1^1) \in H^1(P^1; \mathbb{F}_2) = H^1(S^1; \mathbb{F}_2) = \mathbb{F}_2$.

This sequence of cohomology classes is called the Stiefel-Whitney Classes.

Recall: γ_1^1 for a mobius strip is the zero section, i.e. S^1.

Milnor-Stasheff says naturality a bit differently. Recall: If

$$\begin{tikzcd} E(\eta) \arrow[r, "\text{iso/fibers}"] \arrow[d] & E(\xi) \arrow[d] \\ B(\eta) \arrow[r, "f"] & B(\xi) \end{tikzcd}$$

then $\eta = f^*\xi, \mathrm{w}_i(\eta) = f^*\,\mathrm{w}_i(\xi)$.

Note: axioms 1 and 2 says w_i are *characteristic classes*. Characteristic Classes are cohomology classes respecting naturality. Meaning they respect nontriviality of bundles. Just like homology 'classifies' upto homotopy in a sense, we need characteristic classes to capture the 'twists' in a vector bundle.

Axiom 1 and 2 implies:

Proposition 11 (1)**.** $\xi \cong \eta \implies \mathrm{w}_i(\xi) = \mathrm{w}_i(\eta)$.

Recall that vector bundles are isomorphic if:

$$\begin{tikzcd} E(\xi) \arrow[rr, "\cong"] \arrow[dr] & & E(\eta) \arrow[dl] \\ & B & \end{tikzcd}$$

Proof. $f = \mathrm{id}$. □

Proposition 12 (2)**.** $\mathrm{w}_i(\epsilon_B^n) = 0$ for $i > 0$.

Proof.

$$\begin{tikzcd} B \times \mathbb{R}^n \arrow[r] \arrow[d] & \mathbb{R}^n \arrow[d] \\ B \arrow[r, "c"] & \mathrm{pt} \end{tikzcd}$$

$\mathrm{w}_i(\epsilon^n_B) = \mathrm{w}_i(c^*\epsilon^n_{\mathrm{pt}}) = c^* \mathrm{w}_i(\epsilon^n_{\mathrm{pt}}) \in H^i(\mathrm{pt}; \mathbb{F}_2) = 0.$

Thus, nontrivial Stiefel-Whitney Class implies nontrivial bundle.

□

Proposition 13 (3)**.** If ϵ trivial then $\mathrm{w}_i(\epsilon \oplus \eta) = \mathrm{w}_i(\eta)$. In other words, w_i stable characteristic classes.

Proposition 14 (4)**.** If ξ is an n-plane bundle with k linearly independent sections, then k of them vanishes:

$$\mathrm{w}_{n-k+1}(\xi) = \cdots = \mathrm{w}_{n-1}(\xi) = \mathrm{w}_n(\xi) = 0$$

Most interesting case is $k = 1$ contrapositive.

$\mathrm{w}_n(\xi) \neq 0 \implies \nexists$ nowhere zero section. Hairy ball theorem!

eg for n odd there exists a nowhere zero section of the tangent bundle TS^n. Therefore, $\mathrm{w}_n(TS^n) = 0$.

Since n is odd $n + 1$ is even, and we can switch the coordiantes in pairs:

$$\underline{x} = (x_1, \cdots, x_n) \mapsto (\underline{x}, -x_2, x_1, \cdots, -x_{n+1}, x_n) \in TS^n \subset S^n \times \mathbb{R}^{n+1}$$

$\mathrm{w}_4(T\mathbb{C}P^2) \neq 0$, $\nexists$ nowhere vanishing vector field on $\mathbb{C}P^2$.

If M^n is a closed n-manifold then $\mathrm{w}_n(TM^n) \equiv \xi(M) \pmod 2$.

Proof. The condition of k linearly independent section is equivalent to existence of a subbundle $\epsilon^k_B \subset \xi$.

Case 1: Suppose ξ has a metric.

Then $\xi = \epsilon^k_B \oplus (\epsilon^k_B)^\perp$.

$\mathrm{w}_i(\xi) = \mathrm{w}_i(\epsilon^{k\perp}_B)$ by proposition 3. Note that $\epsilon^{k\perp}_B$ is a $n - k$ bundle, axiom 1 implies the statement.

Case 2: B is a CW complex so B is paracompact which implies ξ has a metric.

General case: suppose $\begin{matrix} E(\xi) \\ \downarrow \\ B \end{matrix}$. Then $\exists$ CW-approximation $B' \to B$ where B' is a CW complex which is isomorphism on π_* which is isomorphism in homology and cohomology. This reduces to case 2.

□

Friday, 9/19/2025

Recap: Stiefel-Whitney-Classes:

Suppose we have an n-plane bundle $\begin{pmatrix} \mathbb{R}^n & \to & E \\ & & \downarrow \\ & & B \end{pmatrix}$

Then $\mathrm{w}_i\, E = \mathrm{w}_i(\xi) \in H^i(B; \mathbb{F}_2)$.

We have some axioms:

1) $\mathrm{w}_0(\xi) = 1, \mathrm{w}_i(\xi) = 0$ for $i > n$

2) Naturality: if we have $f : B' \to B$ then $\mathrm{w}_i(f^*\xi) = f^* \mathrm{w}_i(\xi) \in H^i(B'; \mathbb{F}_2)$.

   One way to rephrase it is as follows: f^*E is the pullback bundle in the following:

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

   Another way: if we have a bundle map:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

   which is an isomorphism on the fibers, then $f^*E \cong E'$. We have $E' \to B'$ which is equal to $f^*\xi$.

   In Milnor-Stasheff, if we have:

$$\begin{array}{ccc} E(\eta) & \longrightarrow & E(\xi) \\ \downarrow & & \downarrow \\ B(\eta) & \longrightarrow & B(\xi) \end{array}$$

   $\eta \to \xi$ in this case $\mathrm{w}_i(\eta) = f^* \mathrm{w}_i(\xi)$.

   Note that properties 1 and 2 are called characteristic class on a bundle.

3) Whitney Sum formula: $\mathrm{w}_k(\xi \oplus \eta) = \sum_{i+j=k} \mathrm{w}_i(\xi) \mathrm{w}_i(\eta)$

4) $\mathrm{w}_1(\gamma_1^1) \neq 0$.

Recall proposition 3: if ϵ trivial then $\mathrm{w}_i(\epsilon \oplus \eta) = \mathrm{w}_i(\eta)$.

Proposition 4: *obstruction to sections*: If ξ has k-linearly independent sections then the top k Stiefel-Whitney Classes vanish.

Whitney Sum Inverses

Definition. Suppose $\xi \oplus \eta = \epsilon^N$. Then ξ and η are whitney sum inverses of each other.

Example: Normal bundle and tangent bundle.

Fact: $\dim B < \infty$ implies every bundle has an inverse.

Observation: $\mathrm{w}_*(\xi)$ can be computed in terms of $\mathrm{w}_*(\eta)$.

$$0 = \mathrm{w}_1(\xi \oplus \eta) = \mathrm{w}_1(\xi) + \mathrm{w}_1(\eta) \implies \mathrm{w}_1(\xi) = \mathrm{w}_1(\eta)$$

$$0 = \mathrm{w}_2(\xi \oplus \eta) = \mathrm{w}_2(\xi) + \mathrm{w}_1(\xi)\,\mathrm{w}_1(\eta) + \mathrm{w}_2(\eta) \implies \mathrm{w}_2(\xi) = \mathrm{w}_1(\eta)^2 + \mathrm{w}_2(\eta)$$

In Milnor Stasheff, they define a new ring:

$$H^{\prod}(B;\mathbb{F}_2) = \prod_i H^i(B;\mathbb{F}_2)$$

This allows us to take infinite series:

$$\mathrm{w}(\xi) = 1 + \mathrm{w}_1\,\xi + \mathrm{w}_2\,\xi + \cdots \in H^{\prod}(B;\mathbb{F}_2)$$

Then we can rephrase the Whitney sum theorem as follows:

$\mathrm{w}(\xi \oplus \eta) = \mathrm{w}(\xi) \cup w(\eta)$.

Lemma 15. $\left\{1 + a_1 + a_2 + \cdots \in H^{\prod}(B;\mathbb{F}_2) \mid a_i \in H^i(B;\mathbb{F}_2)\right\}$

Proof. Due to 'Euler':

$(1 + a_1 + a_2 + \cdots)^{-1} = \frac{1}{1+(a_1+a_2+\cdots)}$

$= 1 + (a_1 + a_2 + \cdots) + (a_1 + a_2 + \cdots)^2 + (a_1 + a_2 + \cdots)^3$

$= 1 + a_1 + (a_2 + a_1^2) + (a_3 + a_1^3) + \cdots$

□

Notation: Suppose $\mathrm{w}(\xi) \in H^{\prod}(B;\mathbb{F}_2)$ then we can have the formal multiplicative inverse: $\overline{\mathrm{w}}(\xi) \in H^{\prod}(B;\mathbb{F}_2)$ so that $\mathrm{w}(\xi)\overline{\mathrm{w}}(\xi) = 1$

This gives us the following observation: $\xi \oplus \eta = \epsilon^N$ gives us $\mathrm{w}(\xi)\,\mathrm{w}(\eta) = 1 \implies \mathrm{w}(\xi) = \overline{\mathrm{w}}(\eta)$.

eg $H^*(\mathbb{P}^\infty;\mathbb{F}_2) = \mathbb{F}_2[a]$ then we have canonical line bundle γ^1 then $\mathrm{w}(\gamma^1) = 1 + a$ so $(1+a)^{-1} = 1 + a + a^2 + \cdots$ which has infinitely many terms so the inverse might not exist! The line bundle doesn't have any whitney sum inverse.

Theorem 16 (Whitney Duality Theorem)**.** Let $M^n \subset \mathbb{R}^N$ be a smooth manifold. Then,

$$\mathrm{w}_i(TM) = \overline{\mathrm{w}}_i\left(\nu(M \hookrightarrow \mathbb{R}^N)\right)$$

Proof. $(TM \oplus \nu(M \hookrightarrow \mathbb{R}^N)) = T\mathbb{R}^N\big|_M$ □

Lemma 17. Suppose we have a closed codim 1 manifold: $M^n \subset \mathbb{R}^{n+1}$. Then $\mathrm{w}(TM) = 1$.

So Stiefel-Whitney Classes give an obstruction to submanifolds of codimension 1.

Proof. $TM \oplus \nu(M \hookrightarrow \mathbb{R}^{n+1})$ is trivial, $\nu(M \hookrightarrow \mathbb{R}^{n+1})$ gives nowhere zero section. □

Corollary 18. Non-orientable submanifolds must have codimension at least 2.

Recall $P^n = \mathbb{R}P^n = S^n/x \sim -x = \frac{S^n_+}{x \sim -x \text{ when } x \in S^{n-1}}$ = lines in $\mathbb{R}^{n+1}$ through 0.

P^n is a CW complex via the pushout:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & P^{n-1} \\ \big\downarrow & & \big\downarrow \\ D^n & \longrightarrow & P^n \end{array}$$

$P^0 \subset P^1 \subset \cdots \subset P^n$ is the skeleton.

Essentially $P^n = e^0 \cup e^1 \cup \cdots \cup e^n$ with $e^i \cong \overset{\circ}{D}{}^i$.

Cellular chain complex:

$$C_\bullet(P^n; \mathbb{F}_2) = \mathbb{F}_2 \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{F}_2$$

Cochain complex:

$$C^\bullet(P^n; \mathbb{F}_2) = \mathbb{F}_2 \leftarrow \cdots \leftarrow \mathbb{F}_2$$

$H_*(P^n; \mathbb{F}_2) = H^*(P; \mathbb{F}_2) = \{\mathbb{F}_2 : x \leq n\}$

Next: $H^*(P^n; \mathbb{F}_2) = \frac{\mathbb{F}_2[a]}{a^{n+1}}$ truncated polynomial ring.

Monday, 9/22/2025

We do some computations today.

Recall: $P^n = S^n / x \sim -x = \underbrace{e^0 \cup e^1 \cup \cdots}_{P^{n-1}} \cup e^n$

Then $H^*(P^n; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & \text{if } * \leq n; \\ 0, & \text{otherwise.} \end{cases}$

Let $0 \neq a \in H^1(P^n; \mathbb{F}_2)$.

Theorem 19. $H^*(P^n; \mathbb{F}_2) = \frac{\mathbb{F}_2[a]}{a^{n+1}}$, truncated polynomial ring.

Proof. Induction on n and Poincaré Duality.

It is true for $n = 1$.

Now suppose it is true for $n - 1$.

We have injection $i : P^{n-1} \hookrightarrow P^n$. Thus i^* is a ring map isomorphism on dimension $\leq n - 1$.

Thus $a, a^2, \cdots, a^{n-1}$ non-zero.

Question: do we have $a^n \neq 0$?

We use Poincaré Duality to prove that.

Suppose $[P^n] \in H_n(P^n; \mathbb{F}_2) \neq 0$.

Then we have: $\cap[P^n] : H^{n-1}(P^n;\mathbb{F}_2) \xrightarrow{\sim} H_1(P^n;\mathbb{F}_2)$.

Then $\langle a^n, [P^n]\rangle = \langle a^{n-1}, a \cap [P^n]\rangle \neq 0$ since UCT implies:

$H^{n-1}(P^n;\mathbb{F}_2) \xrightarrow{\approx} \mathrm{Hom}(H_{n-1}(P^n;\mathbb{F}_2),\mathbb{F}_2)$ by $\beta \mapsto (b \mapsto \langle \beta, b\rangle)$ and both a^{n-1} and $a \cap [P^n]$ are nonzero. □

Now we can look at SW classses of γ_n^1 and TP^n.

Proposition 20. $\mathrm{w}(\gamma_n^1) = 1 + a \in H^*(P^n;\mathbb{F}_2)$.

Proof. True for $n = 1$ by axiom 4.

Now consider restriction: $\gamma_n^1\big|_{P^1} = \gamma_1^1$.

By the axiom we have $1 + a = \mathrm{w}(\gamma_1^1) = i^* \mathrm{w}(\gamma_n^1)$. □

Now let $\gamma = \gamma_n^1 = \{\{([x], v)\} \mid v \in \mathbb{R}x\} \subset P^n \times \mathbb{R}^{n+1}$ be the tautological line bundle.

$\gamma \subset \epsilon_{P^n}^{n+1} \implies \gamma \oplus \gamma^\perp = \epsilon^{n+1}$.

Therefore, $\mathrm{w}(\gamma^\perp) = \overline{\mathrm{w}}(\gamma) = (1+a)^{-1} = 1 + a + \cdots + a^n \in H^*(P^n;\mathbb{F}_2)$.

Thus $\gamma^\perp$ has no nonzero sections.

Corollary 21. γ_∞^1 over P^∞ has no W.SI.

Question: $\mathrm{w}(TP^n) =?$

Recall: $G \curvearrowright X$ then orbit space $X/G = X/x \sim gx, S^n/C_2 = P^n$.

Theorem 22. i) $TP^n \oplus \epsilon^1 = \underbrace{\gamma \oplus \cdots \oplus \gamma}_{n+1}$.

ii) $\mathrm{w}(TP^n) = (1+a)^{n+1} = \sum_{j=0}^n \binom{n+1}{j} a^j \in H^*(P^n;\mathbb{F}_2)$

Proof. Apply the antipodal map to:

$$TS^n \oplus \nu = \epsilon^{n+1} = \epsilon^1 \oplus \cdots \oplus \epsilon^1 \tag{$*$}$$

To get the following:

$$TP^n \oplus \epsilon = \gamma \oplus \cdots \gamma \tag{$**$}$$

where $C_2 \curvearrowright S^n \times \mathbb{R}^{n+1}$ by $(x, v) \mapsto (-x, -v)$.

Note: $TP^n = (TS^n)/C_2$ since S^n is a covering space of TP^n.

Note: $\nu(S^n \hookrightarrow \mathbb{R}^{n+1}) \cong \epsilon_{S^n}^1$

Note: $\nu(S^n \hookrightarrow \mathbb{R}^{n+1})/C_2 \cong \epsilon_{P^n}^1$

Note: $\epsilon_{S^n}^1/C_2 \cong \gamma$ since $\frac{S^n \times \mathbb{R}}{C_2} \cong E(\gamma)$ by $[(x,t)] \mapsto ([x], tx)$

This proves $(**)$.

Now we prove i $\implies$ ii.

$\mathrm{w}(P^n) = \mathrm{w}(TP^n \oplus \epsilon) = \mathrm{w}((n+1)\gamma) = \mathrm{w}(\gamma)^{n+1} = (1+a)^{n+1}$

□

MS shows $TP^n \cong \mathrm{Hom}(\gamma, \gamma^\perp)$.

Parallelizable Manifolds

Definition. A manifold M^n is *parallelizable* if $TM^n = \epsilon^n_M$ [i.e. if there exists n linearly independent vector fields]

eg S^{2n} is not parallelizable via the hairy ball theorem.

Lie Groups are parallelizable: note that T_eG^n has basis $e_1, \cdots, e_n$, and for $g \in G$ we have $\ell_g : G \to G$ given by $h \mapsto gh$.

We then have $g \mapsto (d\ell_{g*})(e_i)$ giving n linearly independent vector fields.

Thus, $\mathrm{w}_i(TM^n) \neq 0$ for $i > 0$ implies M is not a lie group.

$S^0, S^1, S^3, P^0, P^1, P^3 (= \mathrm{SO}(3))$ are lie groups.

Wednesday, 9/24/2025

Corollary 23 (4.6i)**.** $\mathrm{w}_n(P^n) \neq 0 \iff n$ even.

(ii). $\mathrm{w}(P^n) = 1 \iff n+1 = 2^r$

Corollary 24. n even implies P^n has no nowhere zero vector field.

P^n parallelizable [i.e. TP^n trivial] implies $n = 2^r - 1$.

Proof. 4.6i: $\mathrm{w}_n(P^n) \neq 0 \iff \binom{n+1}{n} a^n \neq 0 \iff n+1 \neq 0 \iff n+1$ odd.

4.6ii: $\mathrm{w}(P^{2^r-1}) = (1+a)^{2^r} = 1 + a^{2^r} = 1$ gives one direction. For other direction, if $n+1 = 2^r m$ for odd $m > 1$ then $\mathrm{w}(P^n) = (1+a)^{2^r m} = (1+a^{2^r})^m = 1 + ma^{2^r} + \cdots$.

□

Theorem 25 (4.7 Stiefel)**.** Suppose $\exists$ bilinear map $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ without zero divisor [meaning $p(x, y) = 0 \implies x = 0$ or $y = 0$].

Then P^{n-1} is parallelizabl [thus $n = 2^r$].

e.g. $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Theorem by Adams states $n = 1, 2, 4, 8$.

Proof. Let $\{b_1, \cdots, b_n\}$ be basis for $\mathbb{R}^n$. Define v_i:

$$\begin{tikzcd} \mathbb{R}^n & \mathbb{R}^n \arrow[l, "p(-,b_1)"'] \arrow[r, "p(-,b_i)"] & \mathbb{R}^n \arrow[from=1-1, to=1-3, bend right, "v_i"'] \end{tikzcd}$$

Then $x \neq 0 \implies p(x, b_1), \cdots, p(x, b_n)$ are linearly independent, thus $v_1(x), \cdots, v_n(x)$ linearly independent.

Note that $v_1(x) = x$.

Define linearly independent sections $s_2, \cdots, s_n$ of TP^{n-1}.

$s_i[x] = [x, \mathrm{pr}_{(\mathbb{R}x)^\perp}(v_i(x))] \in TP^{n-1} = (TS^{n-1})/C_2$. □

Stiefel-Whitney Numbers

We want to prove the following theorem:

Theorem 26. A closed manifold is a boundary $\iff$ Stiefel-Whitney numbers are all zero.

We need to talk about first fundamental class.

If M^n is a closed connected manifold [since we have $\mathbb{F}_2$ coefficient we don't worry about orientation] then the fundamental class $[M] \in H_n(M; \mathbb{F}_2) \cong H^0(M; \mathbb{F}_2) = \mathbb{F}_2$.

We dont really need connectedness. If $M^n = M_1 \sqcup \cdots \sqcup M_k$ where each M_j are connected then the fundamental class $[M] = i_{1*}[M_1] + \cdots + i_{k*}[M_k] \in H_n(M; \mathbb{F}_2) = \mathbb{F}_2^k$.

Definition. A partitition of n is $r_1, \cdots, r_n \in \mathbb{Z}_{\geq 0}$ such that $r_1 + 2r_2 + 3r_3 + \cdots + nr_n = n$.

Let $\Pi(n) =$ set of partitions of n.

For example, $\Pi(4) = \{(0,0,0,1), (0,2,0,0), (1,0,1,0), (2,1,0,0), (4,0,0,0)\}$

Definition (Stiefel-Whitney Number)**.** Given $(r_i) \in \Pi(n)$ the Stiefel-Whitney Number is defined by:

$$\mathrm{w}_1^{r_1} \cdots \mathrm{w_n}^{r_n}[M] := \langle \mathrm{w}_1(TM)^{r_1} \cup \cdots \mathrm{w}_n(TM)^{r_n}, [M] \rangle \in \mathbb{F}_2$$

For example we find Stiefel-Whitney numbers of P^2.

$\mathrm{w}(P^2) = \mathrm{w}(TP^2) = (1+a)^3 = 1 + a + a^2$.

$\mathrm{w}_1^2[P^2] = \langle a^2, P^2 \rangle = 1$

$\mathrm{w}_2[P^2] = \langle a^2, P^2 \rangle = 1$

Thus P^2 is not the boundary of a 3-manifold.

We can see this more easily since the characteristic of P^2 is odd.

Friday, 9/26/2025

Homeowork Due Monday.

Ch2: 1 Exercise Ch3: 1 Exercise Ch4: 2 Exercise

Manifolds with Boundary

Classic examples: disk D^n, cylinder $S^{n-1} \times I$

Definition. *Local Model* is the upper half-space $H^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 \geq c\}$.

Definition. Let $M \subset \mathbb{R}^A$. An *n-manifold with boundary* such that $\forall x \in M, \exists$ smooth homeomorphism (parameterization) $h : V \to U$ where $V \subset H^n$ and $x \in U \subset M$ open such that $\forall y \in V, \mathrm{d}h_y : \mathbb{R}^n \to \mathbb{R}^A$ has rank n.

Definition. $\operatorname{Int} M \coloneqq \{x \in M \mid \exists \text{nbhd } U \cong \mathbb{R}^n\}$.

$\partial M \coloneqq M - \operatorname{Int} M$

$M = \partial M \cup \operatorname{Int} M$

$D^n = S^{n-1} \cup \operatorname{Int} D^n$

n-manifold is n-manifold with boundary.

manifold with nonempty interior is not a manifold.

M is an n-manifold with boundary $\implies \operatorname{Int} M$ is a n-manifold and ∂M is a $n-1$ manifold.

$M \simeq \operatorname{Int} M$.

Now consider tangent space:

$$\begin{array}{ccll} \mathbb{R}^n & \to & TM & = \{(x,v) \mid x \in M, v = \gamma'(0), \gamma(0) = x \\ & & \downarrow & , \gamma : [0,\infty) \to M \vee \gamma : (-\infty, 0] \to M\} \\ & & M & \end{array}$$

Then $TM\big|_\partial \cong T\partial M \oplus \epsilon^1$ where ϵ^1 is the outward poinitng normal, the nowhere zero section of $TM\big|_\partial$.

Poincaré-Lefschetz Duality

(PL duality).

Theorem 27. $H_n(M, \partial M; \mathbb{F}_2) = \mathbb{F}_2$.

Definition. Fundamental class $[M] \in H_n(M, \partial M; \mathbb{F}_2)$.

Theorem 28 (PL Duality)**.** $\cap[M] : H^i(M, \partial M; \mathbb{F}_2) \xrightarrow{\approx} H_{n-i}(M; \mathbb{F}_2)$.

$\cap[M] : H^i(M, \mathbb{F}_2) \xrightarrow{\approx} H_{n-i}(M, \partial M, \mathbb{F}_2)$.

Exercise: Work this out for D^n.

Furthermore, if we look at the long exact sequence of a pair:

$$H_n(M, \partial M; \mathbb{F}_2) \xrightarrow{\partial} H_{n-1}(\partial M; \mathbb{F}_2) \to H_{n-1}(M; \mathbb{F}_2)$$

then $\partial[M] = [\partial M]$.

Theorem 29 (MS 4.9, Pontryagin)**.** Suppose M is a compact $n+1$-manifold with boundary. Then the Stiefel Whitney numbers of ∂M are 0.

Proof. WLOG M is connected. Let $r_i \in \Pi(n)$ [thus $\sum_i r_i i = n$].

Then $\langle \mathrm{w}_1(T\partial M)^{r_1} \cup \cdots \mathrm{w}_n(T\partial M)^{r_n}, [\partial M](= \partial[M])\rangle$

$= \langle \delta(\mathrm{w}_1(T\partial M)^{r_1} \cdots \mathrm{w}_n(T\partial M)^{r_n}), [M]\rangle$.

Now, recall:

$$H^n(M;\mathbb{F}_2) \xrightarrow{i^*} H^n(\partial M, \mathbb{F}_2) \xrightarrow{\delta} H^{n+1}(M, \partial M; \mathbb{F}_2)$$

WTS: $\mathrm{w}_1(T\partial M)^{r_1} \cdots \mathrm{w}_n(T\partial M)^{r_n} \in \operatorname{im} i^*$.

Note that it is equal to:

$$\mathrm{w}_1\left(i^*(TM)\right)^{r_1} \cdots \mathrm{w}_n(i^*(TM))^{r_n} = i^*\left(\mathrm{w}_1(TM)^{r_1} \cdots \mathrm{w}_n(TM)^{r_n}\right)$$

□

Theorem 30 (P-Thom)**.** A closed n-manifold is the boundary of a compact n-manifold iff all Stiefel Whitney numbers vanish.

Note that all manifolds are boundary of a not necessarily compact manifold, just take $M \times [0, \infty)$

Definition (Bordism Groups)**.** Two closed n-dimenstional manifolds M_1, M_2 are *bordant* if $\exists$ a compact W^{n+1} manifold with boundary such that $\partial W \underset{\text{diff}}{\cong} M_1 \coprod M_2$. W is called the *cobordism.*

Easy exercise: Bordism is an equivalence relation. Canonical example: Pant $\implies S^1 \sim S^1 \coprod S^1$.

One can get a group $\Omega_n^o =$ (bordism classes of closed n-manifold, $\coprod$).

This is called the unoriented bordism group.

Note that $2\Omega_n^o = 0$ since $\partial(M \times I) = M \coprod M$, $-[M] = [M]$.

Theorem 31 (Collar Neighborhood)**.** $\exists$ neighborhood U of ∂W and a diffeomorphism $h : U \xrightarrow{i} \partial W \times [0, \infty)$ such that $h(x, 0)$ for $x \in \partial W$.

Note that Ω_*^o is a graded ring with cartesian product.

n	Ω_o^n	$\Pi(n)$
0	$\mathbb{Z}/2$ pt	1
1	0	1
2	$\mathbb{Z}/2\, P^2$	2
3	0	0
4	$\mathbb{Z}/2 \oplus \mathbb{Z}/2\, P^4, P^2 \times P^2$	5
5	$\mathbb{Z}/2$ Wu-n-manifold $SU(3)/SO(3)$	7

Table 2: Bordism Group Calculations

Theorem 32 (PT Theorem)**.**

$$\Omega_n^o \overset{\mathrm{SW\#}}{\rightarrowtail} (\mathbb{F}_2)^{\Pi(n)}$$

Monday, 9/29/2025

Applications:

Let $M \to \overline{M}$ is a k to 1 covering map with k odd. Then,

$0 = [M] \in \Omega_n^o \iff 0 = [\overline{M}] \in \Omega_n^o$

eg Lens spaces $L(k)$ with k odd are boundaries.

Proof. $H_n(M; \mathbb{F}_2) \xrightarrow[\cong]{\cdot k} H_n(\overline{M}; \mathbb{F}_2)$.

SW numbers of M = SW numbers of $\overline{M}$. □

MS poses the question:

Why is P^{2k-1} a boundary?

Proof. First proof:

We explicitly calculate the SW numbers.

$\mathrm{w}(P^{2k-1}) = (1+a)^{2k} = (1+a^2)^k$.

Thus, for i odd, $\mathrm{w}_i(P^{2k-1}) = 0$.

Thus, since $\sum_i ir_i$ is odd:

Taking mod 2 $\to \sum_{i \text{ odd}} r_i$ is odd so some odd r_i is nonzero. Thus, $\mathrm{w}_1^{r_1} \cdots \mathrm{w}_{2k-1}^{r_{2k-1}}[P^{2k-1}] = 0$.

Second proof:

If $\exists$ free C_2-action on M then M is a boundary.

Proof: $\partial(M \times_{C_2} [1,-1]) = M \times_{C_2} \{-1,1\} = M$.

Or: $\begin{array}{ccc} S^0 & \longrightarrow & M \\ & & \downarrow \\ & & \overline{M} \end{array}$, change fiber D^1 gives us $\begin{array}{ccc} D^1 & \longrightarrow & W = M \times_{C_2} [-1,1] \\ & & \downarrow \\ & & \overline{M} \end{array}$ which gives us $\partial W = M$.

Lens space $L(4)$ with $\pi_1 = C_4$ then covered by P^{2k-1}. □

Conjecture by Farrell/Yau:

Almost flat manifolds are boundaries.

Theorem 33 (Gromov)**.** Almost flat $\iff$ infranil $\overset{def}{\iff}$ $\begin{array}{c} \text{nilmanifold} \\ \downarrow \\ M \end{array}$ finite cover

Nilmanifold is a simply connected lie group modulo a lattice. Example: $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$, lattice is where $*$ are integers.

Theorem 34 (D-Fang)**.** Yes if finite cover is 2^k-to-1.

$N/\Gamma \to M$ if 2^k-to-1 implies $M = \partial W$.

Chapter 5

$\mathbb{R}P^{k-1}$ = lines in $\mathbb{R}^k$. By lines we mean 1-dim spaces through the origin. Easier to think of $\frac{S^{k-1}}{x \sim -x}$ usually.

We have the tautological line bundle given by $E(\gamma) = \{(\text{line, point on line})\} \subset \mathbb{R}P^{k-1} \times \mathbb{R}^k$.

$$\begin{tikzcd} \mathbb{R} \arrow[r] & E(\gamma) \arrow[d] \\ & P^{k-1} \end{tikzcd}$$

Instead of lines we can think about higher dimensional vector spaces through the origin which gives us the Grassmanian.

Grassmanian or Grassmanian Manifold of n-planes in $\mathbb{R}^k$

Notation: $G_n(\mathbb{R}^k)$ is the Grassmanian. Points are n-dim subspaces of $\mathbb{R}^k$.

$X \in G_n(\mathbb{R}^k) \implies X = n$-dim subspaces of $\mathbb{R}^k$.

Example: planes through the origin in $\mathbb{R}^n$.

We have a tautological n-plane bundle $E(\gamma^n) = \{\text{point, point on plane}\}$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & E(\gamma^n) = \{(X, v) \in G_n(\mathbb{R}^k) \times \mathbb{R}^k \mid v \in X\} \arrow[d] \\ & G_n(\mathbb{R}^k) \end{tikzcd}$$

Suppose $M^n \subset \mathbb{R}^k$. Then we have $M \to G_n(\mathbb{R}^k), p \mapsto T_pM$.

We in fact have a bundle map:

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & E(\gamma^n) \arrow[d] \\ M \arrow[r] & G_n(\mathbb{R}^k) \end{tikzcd}$$

$$\begin{tikzcd} (p, v) \arrow[r, mapsto] \arrow[d, mapsto] & (T_{\pi(v)}M, v) \arrow[d, mapsto] \\ p \arrow[r, mapsto] & T_pM \end{tikzcd}$$

We can do the same for the normal bundle.

$$\begin{tikzcd} \nu(M \hookrightarrow \mathbb{R}^k) \arrow[r] \arrow[d] & E(\gamma^{k-n}) \arrow[d] \\ M \arrow[r] & G_{k-n}(\mathbb{R}^k) \end{tikzcd}$$

Topology on $G_n(\mathbb{R}^k)$

We need to find an atlas. What is the dimension?

Definition (Stiefel Manifold)**.** $V_n(\mathbb{R}^k)$ = orthonormal n-frames in $\mathbb{R}^k$

$= \left\{(v_1, \cdots, v_n) \in \mathbb{R}^k \times \cdots \mathbb{R}^k \mid v_i \cdot v_j = \delta_{ij}\right\}$.

This is a closed, bounded subsset of $(\mathbb{R}^k)^n \implies$ it is compact.

Thus this has a topology.

Now, we have an onto map $q : V_n(\mathbb{R}^k) \twoheadrightarrow G_n(\mathbb{R}^k)$ with $q(v_1, \cdots, v_n) = \text{Span}\{v_1, \cdots, v_n\}$.

Give $G_n(\mathbb{R}^k)$ the quotient topology, i.e. $U \subset G_n(\mathbb{R}^k)$ is open iff $q^{-1}U$ is open.

Lemma 35 (5.1)**.** $G_n(\mathbb{R}^k)$ is a compact smooth manifold of dimension $n(k-n)$. Furthermore, there is a diffeomorphism $G_n(\mathbb{R}^k) \to G_{k-n}(\mathbb{R}^k)$ by $X \mapsto X^\perp$.

Wednesday, 10/1/2025

$O(n) \to V_n(\mathbb{R}^k)$ Stiefell, On n

$\downarrow q$

$G_n(\mathbb{R}^k) = n$ planes in $\mathbb{R}^k$, Grassmanian.

$q(v_1, \cdots, v_n) = \text{Span}(v_1, \cdots, v_n)$.

Given $V_n(\mathbb{R}^k) \subset (\mathbb{R}^k)^n$ subspace topology.

We give $G_n(\mathbb{R}^k)$ quotient topology.

Lemma 36. $G_n(\mathbb{R}^k)$ is a compact smooth manifold of $\dim n(k-n)$.

Proof. hausdorff?

$X \in G_n(\mathbb{R}^k)$

$v \in \mathbb{R}^k$

$d(x, v) =^{-1} d(x, v)$

$V_n(\mathbb{R}^k) \xrightarrow{q} G_n(\mathbb{R}^k) \to R$

$d(-, v) \circ q$ continuous.

$d(-, v)$ continuous.

If $X \neq Y$ choose $v \in Y - X$.

Let $d = d(X, v)$.

Separate X and Y by:

$d(-,v)^{-1}(-\infty, \frac{d}{2})$ and $d(-,v)^{-1}(\frac{d}{2}\infty)$ $\square$

Atlas? Euclidean Neighborhoods?

$X \in G_n(\mathbb{R}^k)$

$U = U_X = \{y \in G_n(\mathbb{R}^k) \mid X^\perp = \{0\}\}$ open and dense.

$\Gamma : \mathrm{Hom}(X, X^\perp) \to U$

$f \mapsto \mathrm{graph}(f) \subset \mathbb{R}^k = X \oplus X^\perp (\cong X \times X^\perp)$.

$\mathrm{graph}(f) \coloneqq \{v + f(v) \mid v \in x\}$

$$\begin{tikzcd} U \arrow[r, "\Gamma^{-1}"] \arrow[rr, bend right, "\phi"'] & \mathrm{Hom}(X, X^\perp) \arrow[r, "\cong"] & \mathbb{R}^{n(k-n)} \end{tikzcd}$$

Coordinates show ϕ is homeomorphism.

Atlas $\{(U, \phi)\}$

ANother proof:

$O(k) \curvearrowright G_n(\mathbb{R}^k)$ transitively, $(A, X) \mapsto AX$

Isotopy at $\mathbb{R} \times \{0_{k-n}\}$:

is $O(n) \times O(k-n)$.

Thus $G_n(\mathbb{R}^k) = O(k)/O(n) \times O(k-n)$.

If G is a compact lie group and H is a closed subgroup then G/H is a manifold.

$$\begin{tikzcd} O(n) = \frac{O(n)\times O(n-k)}{O(n-k)} \arrow[r] & V_n(\mathbb{R}^k) \arrow[d] & = & O(k)/O(n-k) \\ & G_n(\mathbb{R}^k) & = & O(k)/O(n)x, O(n-k) \end{tikzcd}$$

Associated $\mathbb{R}^n$ bundle is γ^n.

$E(\gamma^n) = V_n(\mathbb{R}^k) \times_{O(n)} \mathbb{R}^n$.

Friday, 10/3/2025

Lemma 37 (5.2)**.** The tautological bundle is a bundle:

$$\begin{array}{ccc} E(\gamma_k^n) & = & \{(X,v) \mid v \in X\} \subset G_n(\mathbb{R}^k) \times \mathbb{R}^k \\ \big\downarrow{\scriptstyle \pi} & & \\ G_n(\mathbb{R}^k) & & \end{array}$$

is a rank n v.b.

Proof. $\pi^{-1}X$ is a vector space: $(X,v)+(X,w)=(X,v+w), c(X,v)=(X,cv)$.

We also want local triviality. Consider $X \in G_n(\mathbb{R}^k)$. Let $U = \{Y \mid Y \cap X^\perp = 0\}$.

$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[rr, "h"] \arrow[dr] & & \pi^{-1}U \arrow[dl] \\ & U & \end{tikzcd}$$

is a fiberwise isomorphism where h is a homeomorphism.

Then $U \times \mathbb{R}^n \cong U \times X$ by choosing a basis for X. Furthermore, $U \times X \xleftarrow{\Gamma \times \mathrm{id}_X} \mathrm{Hom}(X, X^\perp) \times X$ and $\mathrm{Hom}(X,X^\perp) \times X \to \pi^{-1}U$ by $(f,v) \mapsto (\mathrm{graph}\, f, v + f(v))$. $\square$

Lemma 38 (5.3)**.** Any n-plane bundle ξ over a compact Hausdorff manifold, $\exists$ a bundle map to the tautological bundle $G_n(\mathbb{R}^k)$:

$$\begin{tikzcd} E(\xi) \arrow[r, "\widetilde{c}"] \arrow[d] & E(\gamma_k^n) \arrow[d] \\ B \arrow[r, "c"] & G_n(\mathbb{R}^k) \end{tikzcd}$$

for k large.

So the tautological bundle is final.

Note that we knew this for embedded manifold and tangent bundle:

$$\begin{tikzcd} TM \arrow[d] & \\ M \arrow[r] & G_n(\mathbb{R}^k) \end{tikzcd}$$

$$p \longmapsto T_pM$$

c is called 'classifying group' and γ^n is the universal bundle.

By defintion, a bundle map $\xi \to \gamma_k^n$ is the same as a fiberwise isomorphism:

$$\begin{tikzcd} E(\xi) \arrow[r] \arrow[d] & E(\gamma_k^n) \arrow[d] \\ B \arrow[r] & G_n(\mathbb{R}^k) \end{tikzcd}$$ which is by definition the same as a fiberwise monomorphism $\hat{c} : E(\xi) \to \mathbb{R}^k$.

Let $F_b = \pi^{-1}b$. Then $c(b) = \hat{c}(F_b) \leftarrow\!\shortmid \hat{c}$.

Then $\widetilde{c}(e) = (\hat{c}(F_b), \hat{c}(e))$.

Now we prove lemma 5.3.

Proof. Compact, so choose open cover $U_1, \cdots, U_r$ of B such that $\xi\big|_{U_i}$ is trivial.

Choose open $W_i \subset V_i \subset U_i$ such that $\overline{W}_i \subset V_i, \overline{V}_i \subset U_i$, and $\{W_i\}$ and $\{V_i\}$ still cover B.

Note that $\overline{W}_i$ and $B - V_i$ are disjoint closed sets. Thus $\exists$ continuous $\lambda_i : B \to [0,1]$ such that $\lambda_i(\overline{W}_i) = 1, \lambda_i(B - V_i) = 0$ by Urysohn's lemma.

$\xi\big|_{U_i}$ trivial $\iff$ fiberwise isomorphism $h_i : \pi^{-1}U_i \to \mathbb{R}^n$ by sections $s_j(b) \mapsto e_i$.

Define $\hat{c} : E(\xi) \to \underbrace{\mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n}_{r \text{ times}}$.

$$\hat{c}(e) = (\lambda_1(\pi(e))h_1(e), \cdots, \lambda_r(\pi(e))h_r(e))$$

$\square$

Corollary 39 (Not in MS)**.** Every vector bundle ξ over a compact Hausdorff space B has a whitney sum inverse.

What we need is a finite locally trivial cover. .

Let $\xi = c^*(\xi_k^n)$. Consider $\xi \oplus c^*(\gamma^\perp) = c^*(\gamma \oplus \gamma^\perp) = c^*(\epsilon^k_{G_n(\mathbb{R}^k)})$ which is trivial. $\square$

Contrast this with the fact that γ^1_∞ has no whitney sum inverse.

Comment: $k' \leq k \implies$
$$\begin{tikzcd}
E(\gamma^n_{k'}) \arrow[r, hook] \arrow[d] & E(\gamma^n_k) \arrow[d] \\
G_n(\mathbb{R}^{k'}) \arrow[r, hook] & G_n(\mathbb{R}^k)
\end{tikzcd}$$

Theorem 40. If $f, g : \xi \to \gamma^n_k$ bunndle maps then $f \simeq g : \xi \to \gamma^n_{2k}$.

So "classifying map unique upto homotopy."

Proof. WTS: $\hat{f} \simeq \hat{g} : E(\xi) \to \mathbb{R}^{2k}$ fiberwise monomorphism.

Special case: $\forall e \in E(\xi), \forall \lambda > 0, \hat{f}(e) \neq -\lambda\hat{g}(e)$. $h_t(e) = (1-t)\hat{f}(e) + t\hat{g}(e)$.

General case: define embeddings $d_0, d_1, d_2 : \mathbb{R}^k \to \mathbb{R}^{2k}$.

$d_0(e_i) = e_i, d_1(e_i) = e_{2i-1}, d_2(e_i) = e_{2i}$.

Then $d_0 \circ \hat{f} \simeq d_1 \circ \hat{f} \simeq d_2 \circ \hat{g} \simeq d_0 \circ \hat{g}$. $\square$

Monday, 10/6/2025

Recall lemma 5.3: all vector bundle ξ over compact hausdorff B there exists a bundle map:

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\tilde{c}} & E(\gamma_k^n) \\ \downarrow & & \downarrow \\ B & \xrightarrow{c} & G_n(\mathbb{R}^k) \end{array}$$

for k sufficiently large.

Theorem 41 (5.7)**.** If B is compact Hausdroff and $f, g : \xi \to \gamma_k^n$ are bundle maps, then $f \simeq g : \xi \to \gamma_{2k}^n$.

Recall the proofs required $E(\xi) \to \mathbb{R}^k$ fiber monomorphism.

Theorem 42 (Covering Homotopy Theorem)**.** Slogan: "Homotopy Invariance of Pullback.".

Suppose we have compact hausdorff manifolds and maps:

$$\begin{array}{ccc} & & E(\xi') \\ & & \downarrow \\ B & \xrightarrow{f \simeq g} & B' \end{array}$$

Then $f^*\xi' \cong g^*\xi'$.

We can 'replace k by ∞ and compact hausdorff by paracompact Hausdorff.'

For 5.2, we use $\infty \cdot n = \infty$.

For 5.7, we use $\infty + \infty = \infty$.

5.3, 5.7 and CHT implies: B paracompact Hausdroff implies there is a bijection between homotopy classes $[B, G_n(\mathbb{R}^\infty)]$ and [iso class of n-plane v.b. over B].

$f \mapsto f^*\gamma^n$.

This is why the Grassmanian is a classifying space, it classifies all bundles.

e.g. for sphere $B = S^l$ then $\pi_l(G_n(\mathbb{R}^\infty)) = \dfrac{\left\{\begin{array}{ccc} \mathbb{R}^n & \to & E \\ & & \downarrow \\ & & S^l \end{array}\right\}}{\text{iso}}$.

Let A be an abelian group and $\mathrm{w} \in H^l(G_n(\mathbb{R}^\infty), A)$.

Then we get characteristic class of n-plane bundle over B a CW complex. Recall CW complexes are paracompact Hausdorff!

Thus, in order to get characteristic classes, we only need:

$$\begin{array}{ccc} E(\xi) & \longrightarrow & E(\gamma^n) \\ \downarrow & & \downarrow \\ B & \xrightarrow{c} & G_n(\mathbb{R}^\infty) \end{array}$$

Then the characteristic class is defined to be $\mathrm{w}(\xi) = c^* \mathrm{w} \in H^l(B; A)$.

Then if we have
$$\begin{tikzcd} & E \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$
then $f^* \mathrm{w}(\xi) = \mathrm{w}(f^*\xi)$.

Theorem 43 (Future Theorem)**.** $H^*(G_n(\mathbb{R}^\infty); \mathbb{F}_2) = \mathbb{F}_2[\mathrm{w}_1, \mathrm{w}_2, \cdots, \mathrm{w}_n]$.

For example, for $n = 1$, this theorem states that $H^*(P^\infty, \mathbb{F}_2) = \mathbb{F}_2[a]$.

First we talk about $\mathbb{R}^\infty$ and $G_n(\mathbb{R}^\infty)$. We talk about colimits for that.

Colimit

Consider Category $\mathcal{C}$.

Definition. A *directed system* (ds):

$$X_0 \to X_1 \to X_2 \to X_3 \to \cdots$$

Definition. A *cocone* of a directed system is an object X with maps so that:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[drrr] & X_1 \arrow[r] \arrow[drr] & X_2 \arrow[r] \arrow[dr] & X_3 \arrow[r] \arrow[d] & \cdots \arrow[dl] \\ & & & X & \end{tikzcd}$$

Definition. A *colimit* of a directed system is an *initial cocone*:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[drrr] \arrow[ddrrr] & X_1 \arrow[r] \arrow[drr] \arrow[ddrr] & X_2 \arrow[r] \arrow[dr] \arrow[ddr] & X_3 \arrow[r] \arrow[d] \arrow[dd, bend right] & \cdots \arrow[dl] \arrow[ddl] \\ & & & C \arrow[d, dotted, "\exists!"] & \\ & & & X & \end{tikzcd}$$

Colimits may not exist. If they exist they are unique upto isomorphism. We write $C = \mathrm{colim}_{n\to\infty} X_n$

Colimit is kind of a 'generalized union'.

Colimits are generally 'quotients of coproducts'.

In the category R-mod,

$$\mathrm{colim}_{n\to\infty} X_n = \frac{\bigoplus_n X_n}{\langle X_n - \mathrm{im}(X_n)\rangle}$$

Thus $\mathbb{R}^\infty := \mathrm{colim}_{n\to\infty} \mathbb{R}^n$, if basis $e_1, e_2, e_3, \cdots$ then almost all coordinates are zero: $(a_1, a_2, \cdots, a_n, 0, 0, \cdots)$

In Top or Set,

$$\mathrm{colim}_{n\to\infty} X_n = \frac{\coprod X_n}{X_n \sim \mathrm{im}(X_n)}$$

Then $G_n(\mathbb{R}^\infty) = \operatorname{colim}_{n\to\infty} G_n(\mathbb{R}^k)$ = set of n-planes in $\mathbb{R}^\infty$ with a particular topology. In some sense, it is $\bigcup_k G_n(\mathbb{R}^k)$.

Stiefel Manifolds

Recall: we have Stiefel Manifolds:

$$\begin{array}{ccc} O(n) & \longrightarrow & V_n(\mathbb{R}^\infty) \\ & & \downarrow \\ & & G_n(\mathbb{R}^\infty) \end{array} \qquad \text{orthonormal } n\text{-frames in } \mathbb{R}^\infty$$

$V_n(\mathbb{R}^\infty) \times_{O(n)} \mathbb{R}^n = E(\gamma^n)$.

Theorem 44. $V_n(\mathbb{R}^\infty)$ is contractible. eg for $n = 1$ we have $S^\infty \simeq *$.

We need some facts from algebraic topology:

1) $V_n\mathbb{R}^\infty$ is a CW complex and $V_n\mathbb{R}^k \subset V_n\mathbb{R}^\infty$ are subcomplexes.
2) Whitehead's Theorem: if X is CW then $X \simeq * \iff \pi_* X = 0$.
3) Given fibration $\begin{array}{ccc} F & \to & E \\ & & \downarrow \\ & & B \end{array}$ (e.g. a (G, F)-bundle) there exists long exact sequence:

$$\cdots \to \pi_i F \to \pi_i E \to \pi_i B \to \pi_{i-1} F \to \cdots$$

Now we can prove the theorem:

Proof. $1 \implies \pi_i(V_n(\mathbb{R}^\infty)) = \operatorname{colim}_{k\to\infty} \pi_i(V_n(\mathbb{R}^k))$

$3 \implies$ for $i \leq l, \pi_i O(l) \xrightarrow{\approx} \pi_i O(l+1)$.

$$\begin{array}{ccccc} O(l) & \to & O(l+1) & & A \\ & & \downarrow & & \downarrow \\ & & S^l & & Ae_{l+1} \end{array}$$

Then $\begin{array}{ccccc} O(k-n) & \to & O(k) & & A \\ & & \downarrow & & \\ & & V_n(\mathbb{R}^k) & & Ae_1, \cdots, Ae_n \end{array}$

$\implies i < k - n, \pi_i(V_n\mathbb{R}^k) = 0 \overset{(2)}{\implies}$ the theorem. ∎

Monday, 10/13/2025

Note:

Schubert Symbol: $\sigma = (\sigma_1, \cdots, \sigma_n)$.

$1 \leq \sigma_1 < \cdots < \sigma_n$.

Dimension $d = d(\sigma) = \sum_i \sigma_i - i$

Partition of $d = \sigma - (1, 2, \cdots, n)$.

Recap:

$G_n(\mathbb{R}^\infty) = BGL(n, \mathbb{R})$.

It is a classifying space.

Proof 1: representative object.

Proof 2: $\begin{array}{ccc} O(n) & \longrightarrow & V_n(\mathbb{R}^\infty) \simeq * \\ & & \downarrow \\ & & G_n(\mathbb{R}^\infty) \end{array}$.

Thus $G_n(\mathbb{R}^\infty) = BO(n), BO(n) = BGL(n, \mathbb{R}), O(n) \simeq GL(n, \mathbb{R})$.

Preview of Chapter 6/7:

- Find CW structure on $G_n\mathbb{R}^\infty$.
- Show mod 2 cellular chain complex has zero differentials. [So this is just like $\mathbb{R}P^\infty$].

Then $H_k(G_n(\mathbb{R}^\infty); \mathbb{F}_2) = C_k(G_n\mathbb{R}^\infty) \otimes \mathbb{F}_2 = \mathbb{F}_2^{\#\text{ of } k\text{-cells}}$.

We use the following two definitions of CW-complexes.

Definition (Using Pushouts)**.** A topological space X together with the filtration $\{X^n\}_{n=0}^\infty$ called skeleton, written $(X, \{X^n\}_{n=0}^\infty)$ so that,

$$X^0 \subset X^1 \subset \cdots \subset X = \bigcup_{n=0}^\infty X^n$$

such that,

1) $\forall n, \exists$ pushout diagram:

$$\begin{array}{ccc} \coprod S^{n-1} & \longrightarrow & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod D^n & \longrightarrow & X^n \end{array}$$

2) $X = \operatorname{colim}_{n\to\infty} X^n$.

Definition (Whitehead)**.** Instead of a filtration we have a partition with cells e_α.

Let X be a Hausdorff space. Consider $((X, \{e_\alpha\}))$ so that,

$\{e_\alpha\}$ form partition of X. i.e. $X = \bigcup_\alpha e_\alpha, e_\alpha \cap e_\beta = \varnothing$ so that,

1) $\forall\alpha, \exists$ characteristic map $\chi_\alpha : D^n \to \overline{e_\alpha}$ such that $\chi_\alpha\big|_{\overset{\circ}{D^n}} : \overset{\circ}{D^n} \xrightarrow{\approx} e_\alpha$ homeomorphism.

2) $\chi_\alpha(S^{n-1}) \subset$ finite union of $n-1$ cells.

3) $B \subset X$ closed $\iff \forall\alpha, B \cap \overline{e_\alpha}$ closed in $\overline{e_\alpha}$.

We can get the skeleton from the cells in the following way: $X^n = \bigcup_{\dim e_\alpha \leq n} e_\alpha$.

Also note $2'$ alternate: $\overline{e_\alpha} - e_\alpha \subset$ finite union of $n-1$ cells.

For skeleton to cell, note that $X^n - X^{n-1}$ is topologically $\coprod_{n\text{-cells}} e_\alpha$.

We want to figure out the CW complex of the Grassmanian. This is connected to combinatorics.

Definition (Schubert Symbol)**.** The cells will be indexed by Schubert Symbol, which will be increasing sequence of integers: $\sigma = (\sigma_1, \cdots, \sigma_n)$ so that $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n$. This will index a 'Schubert cell' of $G_n\mathbb{R}^k$ if $\sigma_n \leq k$:

$$e(\sigma) = \left\{X \in G_n\mathbb{R}^k \mid \forall i, \dim(X \cap \mathbb{R}^{\sigma_i}) = i, \dim(X \cap \mathbb{R}^{\sigma_{i-1}}) = i-1\right\}$$

So we have a dimension jump at $\mathbb{R}^{\sigma_i}$.

$\dim e(\sigma) = \sum_i \sigma_i - i$

Theorem 45 (6.4)**.** $(G_n\mathbb{R}^k, \{e(\sigma)\})$ is a CW complex [note: $1 \leq \sigma_1 < \cdots < \sigma_n \leq k$], and $\dim e(\sigma) = d(\sigma)$.

It also holds for $k = \infty$, i.e. $G_n\mathbb{R}^\infty, \{e(\sigma)\}$ where $1 \leq \sigma_1 < \cdots < \sigma_n$ is a CW complex.

Example: $G_1(\mathbb{R}^3)$. $\sigma = (1), (2), (3)$.

Thus $G_1\mathbb{R}^3 = e^0 \cup e^1 \cup e^2$.

$e^{(1)}$ is the line given by the x-axis.

$e^{(2)}$ is the set of lines through origin in the xy-plane except the x-axis.

$e^{(3)}$ is the set of lines through origin that are not contained in xy-plane.

Now consider $G_2(\mathbb{R}^3).\sigma = (1,2), (1,3), (2,3)$.

$e(1,2)$ is the xy-plane.

$e(1,3)$ are the planes with one basis x-axis, other basis not the y-axis.

$e(2,3)$ are the planes that doesn't contain the x-axis.

Now consider $G_2(\mathbb{R}^4)$. Then $\sigma = (1,2)[d=0], (1,3)[d=1], (1,4)[d=2], (2,3)[d=2], (2,4)[d=3], (3,4)[d=4]$.

σ	$\dim, d = d(\sigma)$	$\sigma - (1, 2, \cdots, n)$
(1)	0	0
(2)	1	1
(3)	2	2
(4)	3	3
(1357)	6	0 1 2 3

Table 3: Schubert Symbol Dimensions

Corollary 46 (6.7)**.** # of d-cells in $G_n\mathbb{R}^k$ = # of paritions of d into at most n integers $\leq k-n$.

Wednesday, 10/15/2025

Chapter 7 assumes existence of SW classes satisfying axioms 1-4.

Abbreviate $G_n = G_n(\mathbb{R}^\infty)$. We have bundles:

$$\begin{array}{ccccc} \mathbb{R}^n & \longrightarrow & E(\gamma^n) & \subset & G_n \times \mathbb{R}^\infty \\ & & \downarrow & & \\ & & G_n & \coloneqq & G_n(\mathbb{R}^\infty) \end{array}$$

Notation: $\mathrm{w}_k \coloneqq \mathrm{w}_k(\gamma^n)$.

$H^*X = H^*(X;\mathbb{F}_2)$. '$\mathbb{F}_2$-coefficients understood'.

Theorem 47 (7.1)**.**

$$H^*G_n = \mathbb{F}_2[w_1, \cdots, \mathrm{w}_n]$$

The free polynomial ring on generators of degs $1, 2, \cdots, n$.

$\iff$ There is no polynomial relationship between them: if p is a polynomial in n variables and $p(\mathrm{w}_1, \cdots, \mathrm{w}_n) = 0$, we must have $p \equiv 0$.

$\iff$ $\mathrm{w}_1, \cdots, \mathrm{w}_n$ are *algebraically independent.*

Lemma 48. Recall γ^1 is the tautological line bundle.

Let $\xi = \underbrace{\gamma^1 \times \cdots \gamma^1}_{n \text{ times}}$.

i) $\mathrm{w}_1(\xi), \cdots, \mathrm{w}_n(\xi)$ are algebraically independent.

ii) $\mathrm{w}_1, \cdots, \mathrm{w}_n$ are algebraically independent.

Proof.

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & E(\xi) = E(\gamma^1) \times \cdots \times E(\gamma^1) \\ & & \downarrow \\ & & \mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty \end{array}$$

$H^*\mathbb{P}^\infty = \mathbb{F}_2[a]$ by Poincaré duality.

Thus $H^*(\mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty) = H^*\mathbb{P}^\infty \otimes_{\mathbb{F}_2} \cdots \otimes_{\mathbb{F}_2} H^*\mathbb{P}^\infty = \mathbb{F}_2[a_1, \cdots, a_n]$ by Künneth Theorem.

By exercise, $\mathrm{w}(\xi) = \mathrm{w}(\pi_1^*\gamma^1 \oplus \cdots \oplus \pi_n^*\gamma^1) = \prod_k \mathrm{w}(\pi_k^*\gamma^1) = (1+a_1)\cdots(1+a_n)$.

Then $\mathrm{w}_k(\xi) = \sigma_k(a_1, \cdots, a_n)$ the k'th elementary symmetric function.

$\sigma_1, \cdots, \sigma_n$ are algebraically independent [Newton].

ii follows from this. We have:

$$
\begin{array}{ccc}
E(\xi) & \longrightarrow & E(\gamma^n) \\
\downarrow & & \downarrow \\
\mathbb{P}^\infty \times \cdots \times P^\infty & \xrightarrow{c} & G_n
\end{array}
$$

Suppose $p(\mathrm{w}_1, \cdots, \mathrm{w}_n) = 0$. Apply c^* to see $p(\mathrm{w}_1(\xi), \cdots, \mathrm{w}_n(\xi)) = 0 \implies p = 0$. $\square$

Now we finally prove theorem 7.1. We need to prove that the polynomials on SW classes generate the cohomology.

Proof. We have:

$$\mathbb{F}_2[\mathrm{w}_1, \cdots, \mathrm{w}_n] \subset H^*(G_n)$$

Let $\mathbb{F}_2[\mathrm{w}_1, \cdots, \mathrm{w}_n]^d$ be the subspace of degree d polynomials on the w's.

$$\mathbb{F}_2[\mathrm{w}_1, \cdots, \mathrm{w}_n]^d \subset H^d(G_n)$$

$H^d(G_n)$ is a *subquotient* of $C^d(G_n)$. Meaning it is quotient of a subgroup / subgroup of a quotient [same thing].

Note that:

$$\dim_{\mathbb{F}_2} \mathbb{F}_2[\mathrm{w}_1, \cdots, \mathrm{w}_n]^d \leq \dim_{\mathbb{F}_2} H^d(G_n) \leq \dim_{\mathbb{F}_2} C^d(G_n)$$

We will show this is an equality.

Note that $\dim_{\mathbb{F}_2} \mathbb{F}_2[\mathrm{w}_1, \cdots, \mathrm{w}_n]^d$ is the number of monomials $\mathrm{w}_1^{r_1} \cdots \mathrm{w}_n^{r_n}$ of degree d, meaning we need $r_1 + 2r_2 + \cdots + nr_n = d$.

$\dim_{\mathbb{F}_2} C^d(G_n)$ is the number of schubert symbols $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n$ of dimension d, meaning $d = \sum_i (\sigma_i - i)$.

We claim they are in bijection as follows:

$$r_n + 1 < r_n + r_{n-1} + 2 < \cdots < r_n + r_{n-1} + \cdots + r_1 - n$$

Thus all three dimensions are equal. Therefore,

$$\mathbb{F}_2[\mathrm{w}_1, \cdots, \mathrm{w}_n] = H^* G_n$$

Furthermore, we can deduce that $\partial \equiv = 0 \pmod 2$ in $C^* G_n$. $\square$

Corollary 49. We have a classifying map:

$$H^*(G_n) \xrightarrow{c^*} H^*(\mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty)$$

$$\mathrm{w}_k \mapsto \sigma_k(a_1, \cdots, a_n)$$

Thus, $H^*(G^n) \cong H^*(\mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty)^{S_n}$

c^* is injective.

Theorem 50 (7.3 Uniqueness)**.** If $\mathrm{w}(\eta) = 1 + \mathrm{w}_1(\eta) + \cdots$ and $\widetilde{\mathrm{w}}(\eta) = 1 + \widetilde{\mathrm{w}}_1(\eta) + \cdots$ satisfying axioms 1-4, then $\mathrm{w} = \widetilde{\mathrm{w}}$

Proof. Step 1: By axiom 4, $\mathrm{w}(\gamma_1^1) = \widetilde{\mathrm{w}}(\gamma_1^1)$.

Step 2: we have

$$\begin{tikzcd} E(\gamma_1^1) \arrow[r] \arrow[d] & E(\gamma^1) \arrow[d] \\ P^1 \arrow[r, "c"] & P^\infty \end{tikzcd}$$

Recall $c^* : H^1\mathbb{P}^\infty \rightarrowtail H^1\mathbb{P}^1$ is an injection so $\mathrm{w}(\gamma^1) = \widetilde{\mathrm{w}}(\gamma^1)$.

Step 3: Set $\xi = \gamma^1 \times \cdots \times \gamma^1$. Then $\mathrm{w}(\xi) = \widetilde{\mathrm{w}}(\xi)$.

To see this, $\xi = \pi_1^*\gamma^1 \oplus \cdots \oplus \pi_*^n\gamma^1$.

$\mathrm{w}(\xi) = \prod_i (1 + a_i) = \widetilde{\mathrm{w}}(\xi)$.

Step 4: $\mathrm{w}(\gamma^n) = \widetilde{\mathrm{w}}(\gamma^n)$.

$$\begin{tikzcd} E(\xi) \arrow[r] \arrow[d] & E(\gamma^n) \arrow[d] \\ \mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty \arrow[r, "c"] & G_n \end{tikzcd}$$

c^* is injective on H^*. $\mathrm{w}(\xi) = \widetilde{\mathrm{w}}(\xi)$ so $c^*\,\mathrm{w}(\xi) = c^*\widetilde{\mathrm{w}}(\xi) \implies \mathrm{w}(\gamma^n) = \widetilde{w}(\gamma^n)$.

Step 5: $\mathrm{w}(\eta) = \widetilde{\mathrm{w}}(\eta)$ when $B(\eta)$ is CW complex.

To see this, just check:

$$\begin{tikzcd} E(\eta) \arrow[r] \arrow[d] & E(\gamma^n) \arrow[d] \\ B(\eta) \arrow[r] & G_n \end{tikzcd}$$

Step 6: $\mathrm{w}(\eta) = \widetilde{\mathrm{w}}(\eta)$ for alll η.

Take CW approximation:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E(\eta) \arrow[d] \\ B \arrow[r] & B(\eta) \end{tikzcd}$$

$\mathrm{w}(E) = \widetilde{\mathrm{w}}(E)$ so $\widetilde{\mathrm{w}}(\eta) = \widetilde{\mathrm{w}}(\eta)$. $\square$

Friday, 10/17/2025

Existence of SW Classes following Thom

Uses two things: Thom isomorphism theorem and Steenrod squares.

$\mathbb{F}_2$-coefficients understood.

Consider a rank n vector bundle
$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & E \arrow[d, "z"] \\ & B \end{tikzcd}.$$

Then we have
$$\begin{tikzcd} \mathbb{R}^n - 0 \arrow[r] & E_0 \arrow[d] & = & E - z(B) \\ & B & & \end{tikzcd}$$

$z(B)$ zero section.

$b \in B, F_b = \pi^{-1}b, F_{b_0} = \pi^{-1}b - \{0\}$.

Remark. $H^*(F_b, F_{b_0}) \cong H^*(\mathbb{R}^n, \mathbb{R}^n - 0) \cong H^*(D^n, S^{n-1}) \cong \widetilde{H}^*(D^n/S^{n-1}) = \begin{cases} \mathbb{F}_2, & \text{if } * = n; \\ 0, & \text{otherwise.} \end{cases}$

Theorem 51 (8.1, Thom)**.** $\exists! u \in H^n(E, E_0)$ such that $\forall b \in B$,

$i_b^* u \neq 0 \in H^n(F_b, F_{b_0}) = \mathbb{F}_2$.

$\forall k \in \mathbb{Z}, H^k E \xrightarrow{\approx} H^{k+n}(E, E_0), x \mapsto x \cup u$ is an isomorphism.

'Every bundle behaves like the trivial bundle'.

Corollary 52. $H^i(E, E_0) = 0$ for $i < n$.

Definition. $u \in H^n(E, E_0)$ Thom class $u = u_E$.

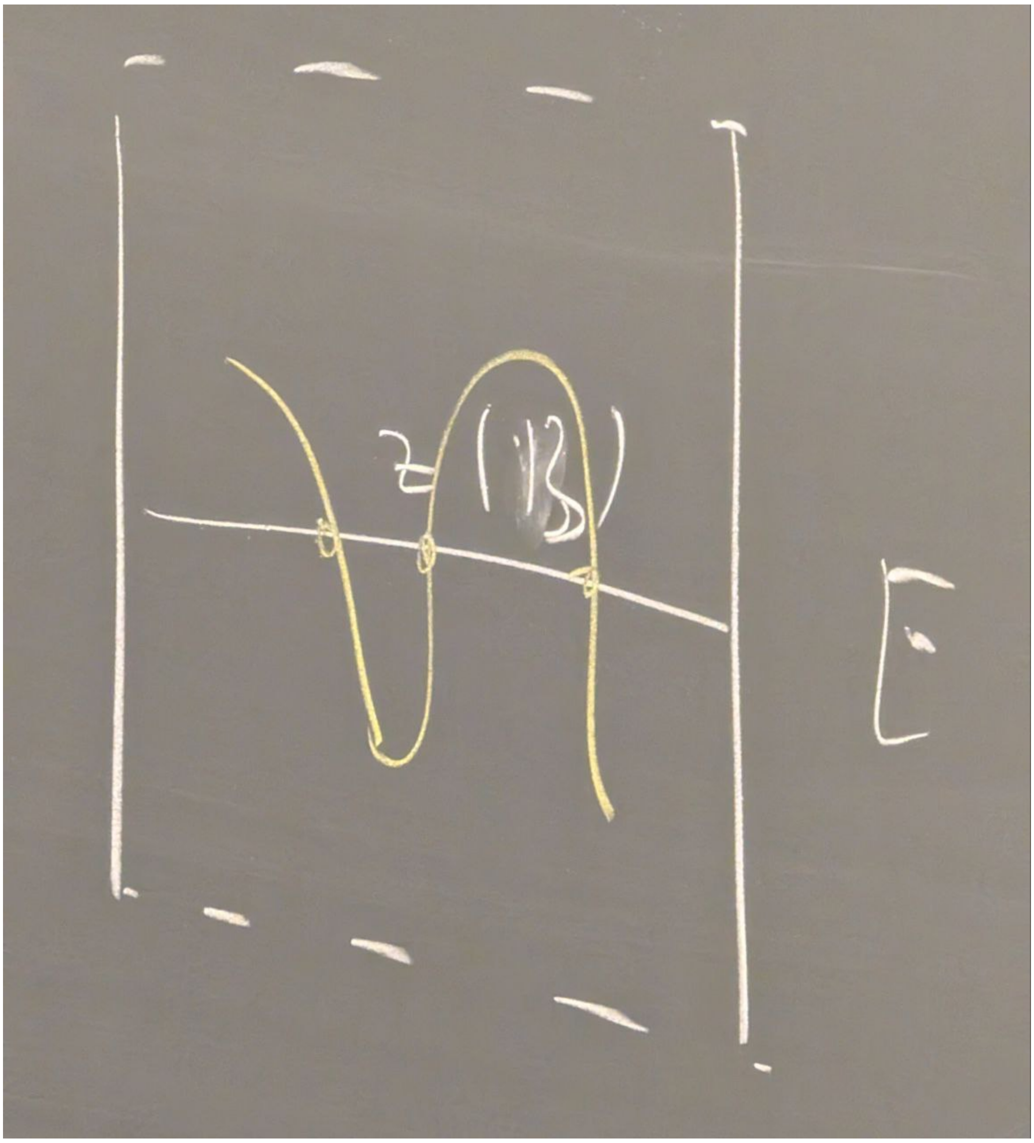
Theorem 53 (Thom Isomorphism)**.** We have the following isomorphism:

$$\phi : H^k B \to H^{k+n}(E, E_0)$$

$$\phi(X) = \pi^* x \cup u$$

Exercise. Prove 8.1 for trivial bundle. [Use Künneth theorem]

What is $\langle u, \text{relative cycle} \rangle$? This is inner product $H^k(E, E_0) \otimes H_k(E, E_0) \to \mathbb{F}_2$. It 'counts' the number of intersections with the zero sections.



Steenrod Squares (Generalizes Cup Products)

Axioms:

1) $\mathrm{Sq}^i : H^n(X,Y) \to H^{n+i}(X,Y)$ homology of abelian groups $\forall n, i \geq 0$.
2) 'naturality' $f : (X,Y) \to (X',Y')$ then $\mathrm{Sq}^i \circ f^* = f^* \mathrm{Sq}^i$.
3) $a \in H^n(X,Y)$.

$$\begin{aligned} \mathrm{Sq}^0 a &= a \\ \mathrm{Sq}^n a &= a \cup a \\ \mathrm{Sq}^i a &= 0 \text{ when } i > n \end{aligned}$$

4) Cartan formula

$$\mathrm{Sq}^k(a \cup b) = \sum_{i+j=n} \mathrm{Sq}^i a \cup \mathrm{Sq}^j b$$

These axioms look like the axioms of SW classes.

Definition (SW Classes, Thom)**.** Let ϕ be the Thom isomorphism. Then,

$$\mathrm{w}_i(\xi) = \phi^{-1}\operatorname{Sq}^i \phi(1) = \phi^{-1}\operatorname{Sq}^i u$$

So, when n is the rank of the bundle,

$$\begin{tikzcd}
u \arrow[rrrr, mapsto] & & & & \operatorname{Sq}^i u \arrow[dd, mapsto, "\phi^{-1}"] \\
& H^n(E, E_0) \arrow[r, "\operatorname{Sq}^i"] & H^{n+i}(E, E_0) & & \\
1 \arrow[uu, mapsto] \arrow[rrrr] & H^0 B \arrow[u] \arrow[r] & H^i(B) \arrow[u] & & \mathrm{w}_i(\xi)
\end{tikzcd}$$

Goal: SW classes satisfy axioms.

Total Steenrod square: $\operatorname{Sq}(a) = a + \operatorname{Sq}^1 a + \operatorname{Sq}^2 a + \cdots + \operatorname{Sq}^n a, a \in H^n(X, Y)$.

Then $\operatorname{Sq} : H^*(X, Y) \to H^*(X, Y), \operatorname{Sq} = 1 + \operatorname{Sq}^1 + \operatorname{Sq}^2 + \cdots$.

Cartan: $\operatorname{Sq}(a \cup b) = \operatorname{Sq}(a) \cup \operatorname{Sq}(b)$.

Axioms for SW classes:

Axiom 1: $\mathrm{w}_0\, \xi = 1, \mathrm{w}_i\, \xi = 0$ for $i > \operatorname{rank} \xi$ follows from 3.

Axiom 2: Naturality:

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\
B \arrow[r, "\overline{f}"] & B'
\end{tikzcd}$$

$f : (E, E_0) \to (E', E_0')$.

Thom class is natural [meaning $f^* u_{E'} = u_E$ since f is isomorphism on fibers].

Thom isomorphism is natural: $f^* \circ \phi_{E'} = \phi_E \circ \overline{f}^*$.

Thus, $\overline{f}^*\, \mathrm{w}_i(\xi') = \overline{f}^* \phi^{-1} \operatorname{Sq}^i \phi(u_{E'}) = \phi_E^{-1} f^* \operatorname{Sq}^i \phi_{E'}(u_{E'}) =$ [some calculations] $= \mathrm{w}_i(\overline{f}^* \xi')$.

Monday, 10/20/2025

Review: $\mathbb{F}_2$-coefficients understood. We have vector bundle $\xi : \mathbb{R}^n \to E \xrightarrow{\pi} B$. We defined $E_0 = E - z(B)$, the complement of the zero section. We defined the Thom class $u = u_E \in H^n(E, E_0)$ so that $i_b^* u \neq 0 \in H^n(F_b, F_{b_0})$ for all $b \in B$.

Thom isomorphism theorem: $\phi_E = \phi : H^*B \to H^{*+n}(E, E_0)$ given by $\phi(x) = (\pi^* X) \cup u_E$ is an isomorphism.

Then we can define SW class of a bundle: $\mathrm{w}_i\, \xi = \phi^{-1} \operatorname{Sq}^i u$.

Recall that $\operatorname{Sq}^i : H^*(E, E_0) \to H^{*+i}(E, E_0)$.

We also have a total version: $\mathrm{w}(\xi) = \phi^{-1} \operatorname{Sq} u_E$ where $\operatorname{Sq} = 1 + \operatorname{Sq}^1 + \operatorname{Sq}^2 + \cdots$

Lemma 54. $\mathrm{w}(\xi \oplus \xi') = \mathrm{w}(\xi) \cup \mathrm{w}(\xi')$.

ALso recall we have the cross product: $H^i X \otimes H^j Y \to H^{i+j}(X \times Y)$ by $a \otimes b \mapsto a \times b$.

This comes from: if we have an n-simplex on $X \times Y$ given by $\sigma : \Delta^n \to X \times Y$, then $(a \times b)(\sigma) = a({}_i(p_X \circ \sigma))b((p_Y \circ \sigma)_j)$ where we have the front i and back j face maps and p_X, p_Y are projections.

Then, $a \times b = (p_X^* a) \cup (p_Y^* b)$ and $a \cup b = \Delta^*(a \times b)$.

Now, suppose we have two bundles $\xi : \mathbb{R}^n \to E \to B$ and $\xi' : \mathbb{R}^{n'} \to E' \to B'$.

Then we can have the cross version of the lemma:

Lemma 55 (X-lemma)**.** $\mathrm{w}(\xi) \times \mathrm{w}(\xi') = \mathrm{w}(\xi \times \xi')$.

Claim: X-lemma implies the lemma.

Proof. $\mathrm{w}(\xi \oplus \xi') = \mathrm{w}(\Delta^*(\xi \times \xi')) = \Delta^* \mathrm{w}(\xi \times \xi') = \Delta^*(\mathrm{w}(\xi) \times \mathrm{w}(\xi')) = w(\xi) \cup w(\xi')$. $\square$

Now we prove the X-lemma.

Proof. $\mathrm{w}(\xi \times \xi') = \phi^{-1}_{E \times E'}(\operatorname{Sq}(u_{E \times E'})) = \phi^{-1}_{E \times E'}(\operatorname{Sq}(u_E \times u_{E'}))$.

Cartan $\implies \operatorname{Sq}(a \cup b) = \operatorname{Sq} \cup \operatorname{Sq} b$, applying Δ^* we see that $\operatorname{Sq}(a \times b)n = \operatorname{Sq} \times \operatorname{Sq} b$.

Thus, $= \Phi^{-1}_{E \times E'}(\operatorname{Sq} u_E \times \operatorname{Sq} u_{E'}) = (\phi_E \times \phi_{E'})^{-1}(\operatorname{Sq} u_E) \times (\operatorname{Sq} u_{E'})$.

$= \mathrm{w}(\xi) \times \mathrm{w}(\xi')$. $\square$

Recall Axiom 4: $\mathrm{w}_1(\gamma_1^1) \neq 0$. We want to prove that.

Proof. Let M be the Möbius strip. Then we have (E, E_0). We also have $(M, \partial M)$. We can collapse the boundary of the möbius strip to a point which gives us $\mathbb{P}^2$. i.e. we have:

$$H^*(E, E_0) \xrightarrow[\text{htpy invariance}]{\approx} H^*(M, \partial M) \xleftarrow[\text{good pair}]{\approx} H^*(M/\partial M, *) \cong H^*(\mathbb{P}^2, *)$$

Recall $E = E(\gamma_1^1) \subset \mathbb{P}^2 \times \mathbb{R}^3, [-1, 1] \times \mathbb{R}/\sim, (x, t) \sim (-x, -t)$.

$u_E \neq 0$ by definition and $H^1(E, E_0) \cong H^1(P^2), u \leftrightarrow a$.

Then, $\operatorname{Sq}^1 a = a \cup a \neq 0 \implies \operatorname{Sq}^1 u \neq 0$.

Thus, $\mathrm{w}_1(\gamma_1^1) = \phi^{-1}(\operatorname{Sq}^1 u_E) \neq 0$.

$\square$

Chapter 9

For this chapter, $\mathbb{Z}$-coefficients understood.

We want to talk about orientation. Let V be a $\dim n$ vector space. Let $V_0 = V - \{0\}$.

Definition. An orientation for V is a generator $\mu_V \in H_n(V, V_0)$.

This corresponds to the linear algebra definition for V.

Orientation of V corresponds to $\frac{\text{ordered bases } (b_1, \ldots, b_n) \text{ for } V}{(b_1,\cdots,b_n)\sim(b_1',\cdots,b_n') \text{ if determinant of change of basis matrix is positive}}$.

Then, the class of $[b_1, \cdots, b_n]$ maps to the orientation in homology given by $\sigma : \Delta^n \to V$ where $\sigma(t_0, \cdots, t_n) = \sum_{i=1}^n (t_i - t_{i-1}) b_i$.

Now suppose $\xi : \mathbb{R}^n \to E \to B$ is a vector bundle.

Definition. Orientation for ξ is an assignment $b \mapsto \mu_{F_b} \in H_n(F_b, F_{b_0}; \mathbb{Z})$ that is 'continuous in b'. Meaning, $\forall b \in B$, $\exists (U, h)$ where $b \in U$ and,

$$\pi^{-1} U \xrightarrow{h} U \times \mathbb{R}^n$$

$\forall x \in U, F_x \to \{x\} \times \mathbb{R}^n \cong \mathbb{R}^n$ is o.p.

If there exists such an orientation we call ξ is orientizable.

Theorem 56 (Thom Isom, 9.1)**.** Let $\xi : \mathbb{R}^n \to E \to B$ be an oriented vector bundle.

i) $\exists! u = u_E \in H^n(E, E_0)$ such that $\forall b, i_b^* u \in H^n(F_b, F_{b_0}) \cong \mathbb{Z}$ is a generator. We call u the Thom class.

ii) $\phi = \phi_E : H^* B \xrightarrow{\approx} H^{*+n}(E, E_0)$ given by $\phi(x) = \pi^* x \cup u$, this is the THom isomorphism.

Corollary 57. $H^k(E, E_0) = 0$ for $k < n$.

$H^n(E, E_0) \cong \mathbb{Z}$ if B is path connected.

e.g. γ_1^1 is path connected.

$H^1(E, E_0; \mathbb{Z}) = H^1(\mathbb{P}^2; \mathbb{Z}) = 0$

Wednesday, 10/22/2025

Let $\xi : \mathbb{R}^n \to E \to B$. Recall that an orientation on ξ is a 'continuous assignment of a point' $b \mapsto \mu_{F_b} \in H_n(F_b, F_{b_0}; \mathbb{Z})$.

Equivalently, continuous assignment of $[b_1, \cdots, b_n]$ an equivalence class of ordered basis of F_b.

$$\begin{array}{ccc} M^n \text{ manifold} & & \text{local homology} \\ \\ \text{cont } x & \longmapsto & \mu_x \in H_n(M, M - x) \cong \mathbb{Z} \end{array}$$

Puzzle: M^n is smooth, orientable on $M \leftrightarrow$ orientation on TM how?

Note that there is $\exp_x : T_xM \to M$ which is a diffeomorphism near x. Patch them up with orientation preserving atlas on M. Meaning, $(M, \mathcal{A})$ where transition maps $\Phi_\beta \circ \Phi_\alpha^{-1}$ are oritentation preserving, meaning their determinant is positive.

$$\begin{tikzcd}
\text{orientation on } M \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{orientation on } TM \arrow[dl, leftrightarrow] \\
& \text{o.p. atlas on } M &
\end{tikzcd}$$

Exercise 12A: $\mathrm{w}_1(\xi) = 0 \iff \xi$ orientable.

Theorem 58. ξ orientable $\iff \mathrm{w}_1(\xi) \in H^1(B; \mathbb{F}_2)$ is 0.

Note that, $\forall n, \exists$ two n-plane bundles over S^1 given by ϵ^n and $\gamma_1^1 \oplus \epsilon^{n-1}$.

$$\text{bundles over } S^1 \xrightarrow{\text{clutching}} \pi_0(\mathrm{GL}_n(\mathbb{R})) \xrightarrow[\det]{\approx} \{\pm 1\}$$

Bundles over I are trivial.

$\xi : \mathbb{R}^n \to E \to B$ homomorphism [orientation character] $\widetilde{\mathrm{w}} : \pi_1 B \to \{\pm 1\}$.

$$\begin{tikzcd}
\epsilon^n \arrow[r] \arrow[d] & E(\gamma) \arrow[d] \\
I \arrow[r, "\exp"] & S^1
\end{tikzcd}$$

$E(\gamma) \cong \frac{I \times \mathbb{R}^n}{(0,v) \sim (1, Av)}$ where $A \in \mathrm{GL}_n(\mathbb{R})$.

$$\widetilde{\mathrm{w}}[\gamma] = \begin{cases} +1, & \text{if } \gamma^* \epsilon \text{ trivial;} \\ -1, & \text{if } \gamma^* \epsilon \text{ non-trivial.} \end{cases}$$

Essentially, given a loop we walk around it to see if my right hand becomes my left hand.

By UCT and Hurewicz theorem,

$$H^1(B; \mathbb{F}_2) \cong \mathrm{Hom}(H_1 B, \mathbb{F}_2) \cong \mathrm{Hom}(\pi_1 B, \{\pm 1\})$$

$$\mathrm{w}_1(\xi) \longleftrightarrow \widetilde{\mathrm{w}}$$

They correspond for γ_1^1 so they correspond for γ.

$\mathbb{P}^\infty = G_1(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ is isomorphism on π_1. Meaning,

$$\pi_1 \mathbb{P}^\infty \to \pi_1 G_n(\mathbb{R}^\infty)$$

by cellular approximation [they have the same 1-skeleton and thus 1-cells. Recall the 1-skeleton contains some Schubert cells with dimension 1. So any path on $G_n(\mathbb{R}^\infty)$ is homotopic to one in $G_1(\mathbb{R}^\infty)$]. It is $1-1$ because of w_1.

Therefore, they correspond for γ^n. Thus, $\mathrm{w} \rightsquigarrow \widetilde{\mathrm{w}}$ for general ξ.

Milnor-Stasheff uses oriented grasmanian $\widetilde{G}_n(\mathbb{R}^\infty)$ to show that $H^1(\widetilde{G}_n(\mathbb{R}^\infty); \mathbb{F}_2) = 0$.

Theorem 59 (Thom Isomorphism Theorem)**.** $\xi : \mathbb{R}^n \to E \to B$. We have a $\mathbb{Z}$-coefficient version and a $\mathbb{F}_2$-coefficient version. In other words, we have a general manifold version and an oriented manifold version.

We can then state the theorem in a fancier way:

$H^*(E, E_0)$ is a free rank 1 module over H^*B with a generator [the thom class] in $\deg n$. This works for both $\mathbb{Z}$ and $\mathbb{F}_2$ coefficients.

The module action is given by the cup product. For $x \in H^*B$ and $a \in H^*(E, E_0)$, we can first take the pullback π^*x of x into H^*E. Then,

$$x \cdot a := \underbrace{(\pi^* x)}_{H^*E} \cup \underbrace{a}_{H^*(E,E_0)} .$$

Then $H^*B \cong H^{*+n}(E, E_0) = H^*B \cup u_E$

Proof 1. We use the Serre Spectral Sequence. We look at the relative fibration:

$$\begin{tikzcd} (F, F_0) \arrow[r] & (E, E_0) \arrow[d] \\ & B \end{tikzcd}$$

We then have the machine that computes the cohomology of the total space in terms of the cohomology of the base with coefficients in the fiber:

$$E^2 = H^p(B; H^q(F, F_0)) \implies H^{p+q}(E, E_0)$$

	0
n	$H^*(F, F_0)$
	0

[Take M622 for more information]. ☐

Friday, 10/24/2025

$\xi : \mathbb{R}^n \to E \to B$

Theorem 60 (Thom Isomorphism)**.** Here $\mathbb{Z}$-coefficient if oriented, else $\mathbb{F}_2$.

Then $H^*(E, E_0)$ is free rank 1 module over H^*B. Generator $u_E \in H^n(E, E_0)$.

Theorem 61 (Thom Isomorphism)**.** $\phi : H^*B \xrightarrow{\approx} H^{*+n}(E, E_0)$, $\phi(y) = \pi^*y \cup u_E$.

Theorem 62 (Thom Isomorphism for Homology)**.**

$$H_*B \xleftarrow{\approx} H_{*+n}(E, E_0)$$

It is given by cap product with the Thom class.

We did first proof via spectral sequences.

Second proof: Mayer-Vietoris.

Proof. Case 1: Trivial bundle.

Here $(E, E_0) = B \times (\mathbb{R}^n, \mathbb{R}^n_0)$. In (E, E_0) we have $H^* = H^*B \otimes H^*(\mathbb{R}^n, \mathbb{R}^n_0)$ is free rank 1 by Künneth theorem.

Case 2: $B = B' \cup B''$ open cover. Assume Thom Isomorphism Theorem holds for $\xi\big|_{B'}, \xi\big|_{B''}$ and $\xi\big|_{B' \cap B''}$.

Write $B^\cap := B' \cap B''$. Let $E^\cap = \pi^{-1}(B^\cap)$ and $E_0^\cap = \pi_0^{-1}(B^\cap)$.

Question: why is this a thom class?

We have the relative Mayer-Vietoris exact sequence:

$$\cdots \longrightarrow H^n(E, E_0) \longrightarrow H^n(E', E_0') \oplus H^n(E_0'', E_0'') \longrightarrow H^n(E^\cap, E_0^\cap) \longrightarrow \cdots$$

$$u \longmapsto (u', u'') \longmapsto 0$$

Thus we must have $u' \mapsto u^\cap \hookleftarrow u''$.

Now we can use a 5-lemma argument:

$$\begin{tikzcd}
H^iB \arrow[r] \arrow[d] & H^iB' \oplus H^iB'' \arrow[r] \arrow[d, "\phi", "\cong"'] & H^iB^\cap \arrow[d, "\phi", "\cong"'] \\
H^{i+n}(E, E_0) \arrow[r] & H^{i+n}(E', E_0') \oplus H^{i+n}(E'', E_0'') \arrow[r] & H^{i+n}(E^\cap, E_0^\cap)
\end{tikzcd}$$

So, $H^iB \xrightarrow{\phi, \cong} H^{i+n}(E, E_0)$.

Case 3: Finite cover $B = B_1 \cup \cdots \cup B_k$ such that $\xi\big|_{B_i}$ is trivial for $\forall i$.

Use induction and Case 2: $(B_1 \cup \cdots \cup B_{k-1}) \cup B_k$.

Thus Thom isomorphism holds if B is compact.

Case 4: General case. Then use limits. Too hard.

$\square$

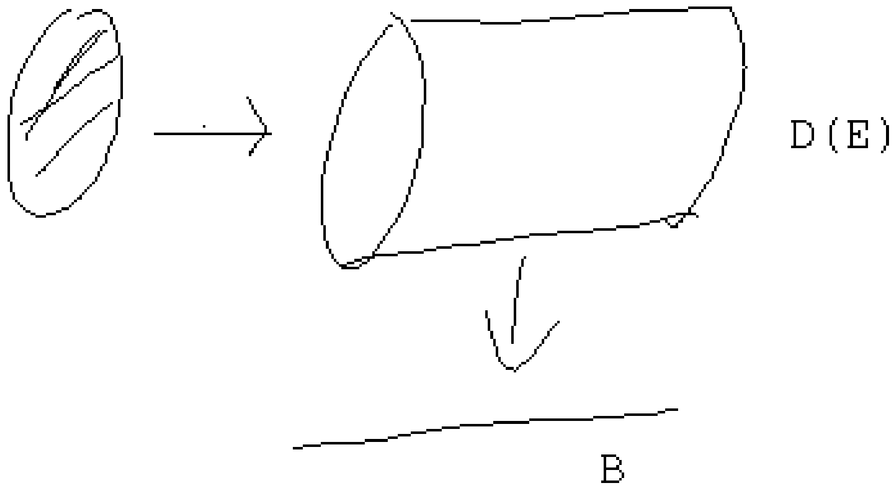
Proof. Third proof. Assume ξ is smooth n-bundle, B is a k-dimensional smooth closed manifold.

We can give ξ a metric $\|\cdot\| : E \to \mathbb{R}$.

Disk bundle $D(E) = \{e \in E \mid \|e\| \leq 1\}$.

$S(E) = \{e \in E \mid \|e\| = 1\}$.

Then $(D(E), S(E)) \to (E, E_0)$ gives isomorphism H_* and H^*



$D(E)$ is a compact manifold with $\partial D(E) = S(E)$.

Let PD_B and PD_{DE} be Poincaré (Lefschetz) duality isomorphims. Define thom class to P-L dual of zero section.

$$u_E \coloneqq PD_{DE}(z_*[B]) \in H^n(DE, SE)$$

Here z is the zero section.

$\phi(y) = \pi^* y \cap u_E \overset{\text{claim}}{=} PD_{DE}(\text{inc}_* PD_B y)$.

For the claim see Bredon's topology and geometry book page 369.

$\square$

Monday, 10/27/2025

Chapter 9

Consider an oriented vector bundle ξ : $\begin{tikzcd} \mathbb{R}^n \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$.

Definition. *Euler Class* $e(\xi) \in H^n(B;\mathbb{Z})$ is the image of the Thom class:

$$\begin{array}{ccccc} H^n(E,E_0) & \longrightarrow & H^nE & \xleftarrow[\cong]{\pi^*} & H^nB \\ \cup & & & & \cup \\ u & & & & e(\xi) \end{array}$$

Three uses:

Proposition 63 (11.12)**.** M^n closed, oriented manifold then,

$$\langle e(TM), [M] \rangle = \chi(M)$$

Where χ is the Euler characteristic.

Proposition 64. Euler class is the first obstruction to the existence of a nowhere zero section.

Thus, $\dim B < n \implies \xi$ has a nowhere zero section.

$\dim B = n, e(\xi) = 0 \implies \xi$ has a nowhere zero section.

Thus, M^n closed, oriented, $\chi(M) = 0 \implies \exists$ nowhere zero vector field.

If $X^n \subset M^{2n}$ closed, oriented then,

$$\langle e(\nu(X \hookrightarrow M)), i_*[X] \rangle = \text{self intersection } \# \text{ of } X$$

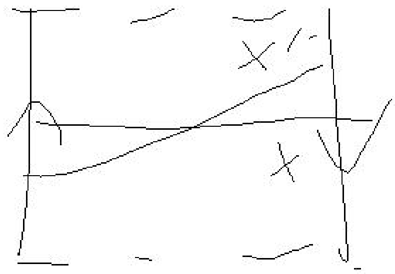
$$= X \cdot X = \langle PD_M[X], [X] \rangle$$

For example, $\langle e(\mathbb{C}P^n \hookrightarrow \mathbb{C}P^2), [\mathbb{C}P^1] \rangle = 1$

Non-oriented clase:

$\langle e(\nu(X \hookrightarrow M)), i_*[X] \rangle = X \cdot X \mod 2$.

Example: consider $M = \mathbb{R}P^2$.



[We pertrube a bit since considering the intersection doesn't really make sense]

Then $e(\xi) \mod 2 = \mathrm{w}_n(\xi)$.

Note that $X \cdot X \mod 2 = X \cdot X' \mod 2$.

Basic Properties, Milnor-Stasheff 9.2

i) 9.2 $e(\xi)$ is natural. i.e. It is a characteristic class. If $f : \xi' \to \xi$ is a bundle map [meaning there is an isomorphism on the fibers]

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\bar{f}} & B \end{array}$$

Then $e(f^*\xi) = f^*e(\xi)$.

ii) 9.3 $\overline{\xi}$ reversing orientation on ξ gives us $e(\overline{\xi}) = -e(\xi)$.

iii) 9.4 n odd $\implies 2e(\xi) = 0, \xi \cong \overline{\xi}$ [oriented vector bundle]. $v \mapsto -v$, then $e(\xi) \underset{9.2}{=} e(\overline{\xi}) = -e(\xi)$.

If M^n is closed and oriented, then $\xi(M^n) = 0, e(TM) = 0$.

$\chi(\mathbb{R}P^2) = 1, e(T\mathbb{R}P^2) \neq 0$

So, if $H^n B$ is torsion free and n is odd, then $e(\xi) = 0$.

If $e(\xi) \neq 0, n$ odd then $e(\xi) \in H^n(B)$ has order 2. Thus there exists a nontrivial torsion summand of $H^n B$.

Question: does there exists unique oriented $\xi : \mathbb{R}^n \to E \to B$ with n odd so that $e(\xi) \neq 0$?

Proposition 65. 9.4.$\frac{1}{2}$: $e(\xi) = \phi^{-1}(u \cup u)$.

Proof. $\phi(e(\xi)) = \pi^* e(\xi) \cup u = u\big|_E \cup u = u \cup u$.

$$
\begin{tikzcd}
u \quad u & u \cup u \\
H^n(E, E_0) \otimes H^n(E, E_0; \mathbb{F}_2) \arrow[r] \arrow[d] & H^{2n}(E, E_0; \mathbb{F}_2) \\
H^n E \otimes H^n(E, E_0; \mathbb{F}_2) \arrow[ur] & \\
u\big|_E \quad u &
\end{tikzcd}
$$

□

Proposition 66. $H^n(B;\mathbb{Z}) \to H^n(B, \mathbb{F}_2)$ has $e(\xi) \mapsto \mathrm{w}_n(\xi)$.

Proof. $e(\xi) \mapsto \phi^{-1}(u \cup u) = \phi^{-1}(\mathrm{Sq}^n\, u) = \mathrm{w}_n(\xi)$ □

Proposition 67 (9.6)**.** a) $e(\xi \times \xi') = e(\xi) \times e(\xi')$.

b) $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$.

Proof. a) Follows from $u_{E \times E'} = u_E \times u_{E'}$.

b) Apply Δ^* to a.

□

Proposition 68 (9.7)**.** If ξ has a nowhere zero section then $e(\xi) = 0$.

Proof. If B is paracompact we can choose a metric. Then, $\xi = \epsilon^1 \oplus (\epsilon^1)^\perp \to e(\xi) = 0 \cup e((\epsilon^1)^\perp) = 0$.

We use CW approximation for general case. □

In general, $e(\xi \oplus \epsilon^1) = 0$. Thus, the Euler class is not stable, in contrast to the Stiefel-Whitney classes, where they are stable w.r.t. 'adding' trivial bundles.

Wednesday, 10/29/2025

Crash Course in Intersection Theory

- Transversality
- Isotopy invariance
- Intersection numbers
- Thom transversality theorem
- Tubular neighborhood theorem
- Explicit PD
- Alg Int # = Gem Int #.

Transversality

Consider submanifolds A, B of M.

Definition. $A \pitchfork B$ [A and B intersect Transfversly] means $\forall x \in A \cap B$, $T_xA + T_xB = T_xM$.

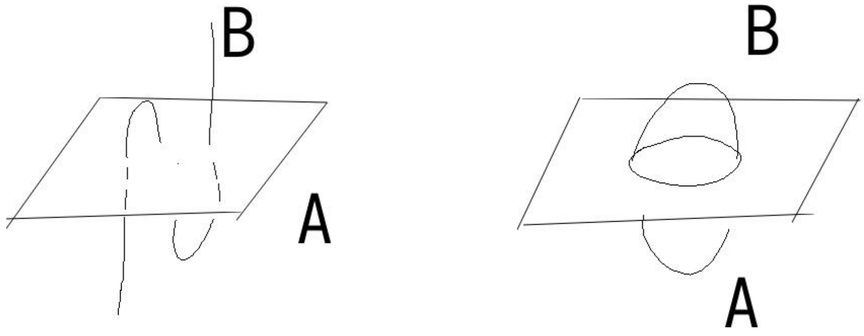


Figure 4: Transverse

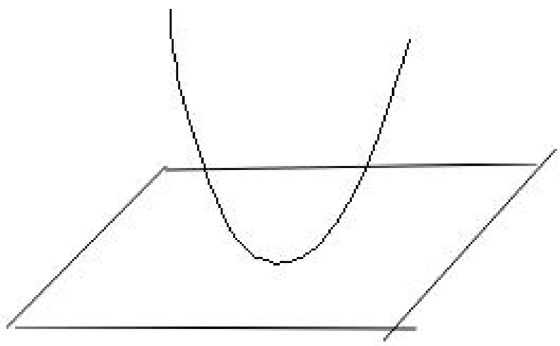


Figure 5: Not transverse

Theorem 69. $A \pitchfork B$. Then,

- $A \cap B$ is a manifold.
- $\nu(A \cap B \hookrightarrow A) \cong \nu(B \hookrightarrow M)\big|_{A \cap B}$.

Furthermore, $\dim A - \dim A \cap B = \dim M - \dim B$.

Recall that $\nu(B \hookrightarrow M) = (TB)^{\perp} \subset TM\big|_B$.

$\nu(B \hookrightarrow M) = \frac{TM|_B}{TB}$.

Theorem 70. All submanifolds A, B where A is isotopic to $A', A' \pitchfork B$.

Slogan: "Transversality is generic". i.e. it is a dense open condition.

We can pertrube A to make it transverse.

Recall isotopy means homotopy through embeddings.

Intersection Numbers

Assume now that $A^n \pitchfork B^k \subset M^{n+k}$.

This implies that $T_xA \oplus T_xB = T_xM$. Assume further that $|A \cap B| < \infty$. e.g. M is compact.

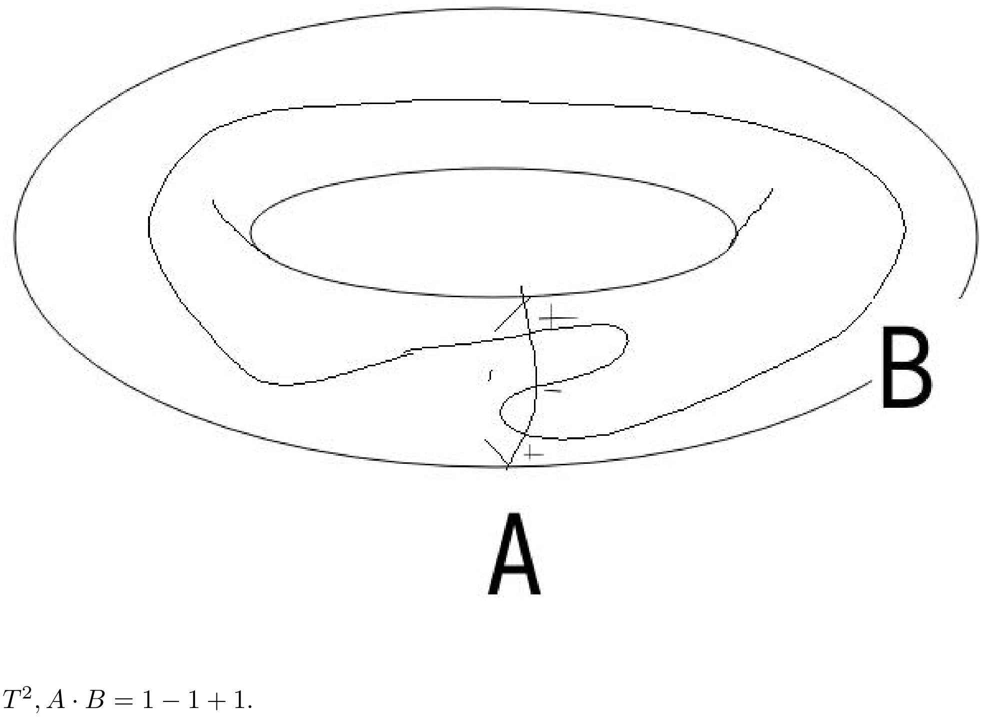
Then we can define the mod 2 interserction number: $|A \cap B| \mod 2$.

Now assume A, B, M are all oriented.

For $x \in A \cap B$ we can define:

$$\epsilon_x = \begin{cases} +1, & \text{if orientation of } T_xA \oplus T_xB \text{ and } T_xM \text{ match;} \\ -1, & \text{otherwise.} \end{cases}$$

$$A \cdot B = \sum_{x \in A \cap B} \epsilon_x \ .$$



There $M = T^2, A \cdot B = 1 - 1 + 1$.

Theorem 71. A, B, M are closed then $A \cdot B$ is isotopy invariant.

First Proof. 'Geometric' □

Second Proof. 'Homological'.

$$A \cdot B = \langle PD_M[A] \cup PD_M[B], [M] \rangle \in \mathbb{Z}$$

$\square$

Observe that A not transverse to B can derive that $A \cdot B \coloneqq A' \cdot B'$.

Consider $M = \mathbb{R}^2, A = S^1$ and $B = I$. Then, $A \cdot B$ is not isotopy invariant.

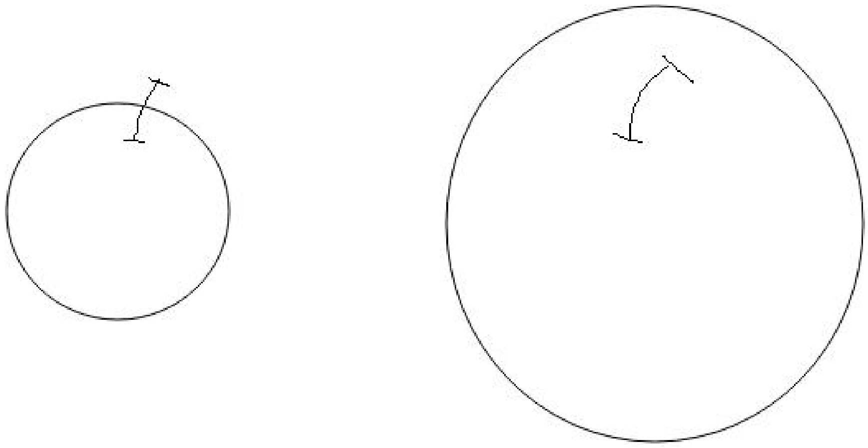


Figure 6: $A \cdot B$ is not isotopy invariant in this case

Suppose $\partial M \neq \varnothing$, submanfiold A of F is called *proper* if $\partial A = A \cap \partial M$.

Theorem 72. If A^n, B^k are proper submanifolds of M^{n+k} where B is closed and A, M are compact, and suppose that $A \pitchfork B$, then $A \cdot B$ is isotopy invariant.

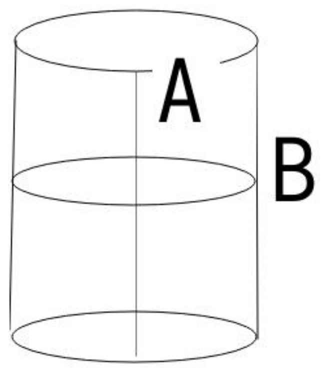


Figure 7: Here $A \cdot B = 1$

Corollary 73. For closed $A, B \subset M$, $A \cdot B$ is isotopy invariant.

Warning: $A, B \subset M$ proper then $A \cdot B$ is not isotopy invariant.

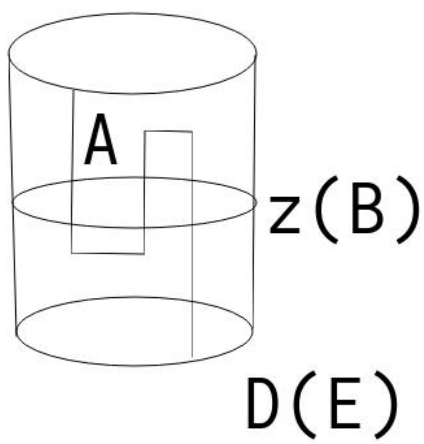
Theorem 74 (Thom Intersection Theorem)**.** Suppose we have a smooth bundle $\xi : \mathbb{R}^n \to E \to B^k$ with metric on ξ and B closed.

Recall that the thom class $u_E = PD_E z_*[B] \in H^n(DE, SE) = H^n(E, E_0)$ where z is a zero section.

If $A^n \subset D(E)$ is a proper compact submanifold, then,

$$A \cdot z(B) = \langle u_E, z_*[B] \rangle \in \begin{cases} \mathbb{Z}, & \text{if oriented;} \\ \mathbb{F}_2, & \text{otherwise.} \end{cases}$$

Proof. After isotopy of A, assume $\exists$ neighborhood of $z(B)$ such that each component $A \cap V$ lies in a fiber.



□

Friday, 10/31/2025

We are moving on to chapter 11.

Let $M^n \subset A^{n+k}$ submanifold.

Theorem 75 (11.1 Tubular Neighborhood Theorem)**.** $\exists$ embedding $\nu(M \hookrightarrow A) \hookrightarrow A$ which is 'identity' on M.

Proof. (When A is compact): Give TM a metric. Consider $\exp : TM \to M$ as follows:

$\exp(v) = \gamma'(1)$ where $\gamma : [0,1] \to M$ geodesic where $\gamma(0) = \pi(v)$ and $\gamma'(c) = v$

We start at the base point and run in the direction of v.

$\exists \epsilon > 0$ such that $\exp \big| \mathring{D}_\epsilon(\nu) \hookrightarrow A$.

Note that $E(\nu) \cong \mathring{D}_\epsilon(v)$ by scaling.

$E(\nu) \hookrightarrow A, (-\epsilon, \epsilon) \cong \mathbb{R}$.

□

Corollary 76 (11.2)**.** If M is losed in A then restriction maps are isomorphisms:

$$H^*(A, A - M) \xrightarrow[\text{excision}]{\cong} H^*(N, N - M) \xrightarrow[TMT]{\cong} H^*(E(\nu), E(\nu)_0)$$

Here N is the tubular neighborhood: $\mathrm{im}(E(\nu) \subset A)$.

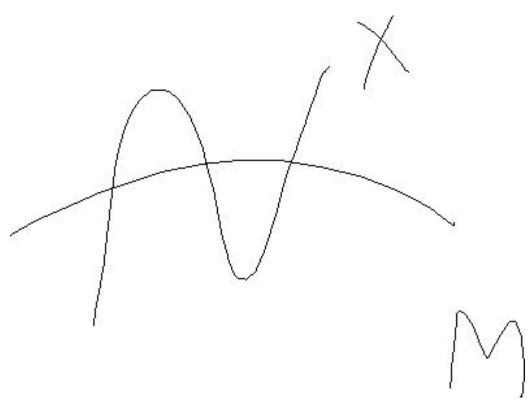
Definition. Thom class $u_M \in H^n(A, A - M)$ maps to u_ν.

$u_M \in H^n(-; \mathbb{F}_2)$

$u_M \in H^n(-; \mathbb{Z})$ if ν is oriented, e.g. A, M are oriented.

Remark. $X \pitchfork M$. $[x] \in H_n(A, A - M)$.

$\langle u_M, [x] \rangle \in M \cdot X$.



Theorem 77 (11.3)**.**

$$H^k(A, A-M) \xrightarrow{i^*} H^k A \xrightarrow{j^*} H^k M$$

a) If M is closed in A then,

$$j^* i^* u_A = \begin{Bmatrix} \mathrm{w}_k(\nu) \\ e(\nu) \end{Bmatrix} \text{ if } \nu \text{ is } \begin{Bmatrix} \\ \text{oriented} \end{Bmatrix}$$

b) If $M \subset A$ but closed in manifolds,

$$i^* u_M = PD[M] \in \begin{cases} H^k(A; \mathbb{F}_2), & \text{if ;} \\ H^k A, & \text{if } A \text{ and } M \text{ both oriented.} \end{cases}$$

Proof. b: explicit Poincaré Duality: Poincaré Dual of submanifold in the image of Thom class of its normal bundle.

$$\begin{tikzcd}
H^n A \arrow[r] & H^k A \\
{[M]} \arrow[u, phantom, sloped, "\in"] \arrow[r] & \operatorname{im} u_M
\end{tikzcd}$$

a: oriented case 'Essentially definition of Euler class'

$$\begin{tikzcd}
 & u_M & \\
H^k(N, N-M) \arrow[d, "TNT"] & H^k(A, A-M) \arrow[l, "\cong"'] \arrow[r, "i^*"] & H^k A \arrow[d, "j^*"] \\
H^k(E(\nu), E(\nu)_0) \arrow[r] & H^k(E(\nu)) \arrow[r, "\cong"] & H^k M \\
u_\nu & & e(U)
\end{tikzcd}$$

□

In the non-oriented case, with $\mathbb{F}_2$-coefficients, need:

$$\begin{tikzcd}
H^k(E(\nu), E(\nu)_0; \mathbb{F}_2) \arrow[r] & H^k(M; \mathbb{F}_2) \\
u_\nu \arrow[r, mapsto] & \mathrm{w}_k(\nu)
\end{tikzcd}$$

[See 95]

Applications:

Corollary 78 (11.3a)**.** $\implies$ Cor 11.4. $M^n \subset \mathbb{R}^{n+k}$ closed subset then,

$0 = \mathrm{w}_k(\nu) = \overline{\mathrm{w}}_k(TM)$.

If $M \subset \mathbb{R}^{n_k}$ is oriented, then $e(\nu) = 0$.

Recall that $\overline{\mathrm{w}}(\xi)\,\mathrm{w}(\xi) = 1$.

$\overline{\mathrm{w}}(\xi) = \mathrm{w}(\xi)^{-1} = \frac{1}{1+(\mathrm{w}_1+\mathrm{w}_2+\cdots)}$

$= 1 + (\mathrm{w}_1+\mathrm{w}_2+\cdots) + (\mathrm{w}_1+\mathrm{w}_2+\cdots)^2 + \cdots$

Recall $M^n \hookrightarrow \mathbb{R}^{n+k}$ immersion implies $\overline{\mathrm{w}}_l(TM) = 0$ for $l > k$.

When $n = 2^l$, $\mathrm{w}(TP^n) = 1 + a + a^n$.

$\mathrm{w}(TP^n) = 1 + a + \cdots + a^n$.

Therefore, $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{2n-2}$.

We can go down one further dimension $\mathbb{R}P^n$ doesn't embed in $\mathbb{R}^{2n-1}$. In particular, $\mathbb{R}P^2 \not\hookrightarrow \mathbb{R}^3$.

Now, consider the open Möbius strip M.

$M \hookrightarrow \mathbb{R}^3$ but $\mathrm{w}_1(TM) \neq 0 \implies \overline{\mathrm{w}}_1(TM) \neq 0$

This means $M \not\hookrightarrow \mathbb{R}^3$ as closed subset.

Monday, 11/3/2025

Chapter 11

Goals: Euler class of a closed manifold integrated over the whole manifold is the Euler characteristic:

$$\langle e(T), [M]\rangle = \chi(M)$$

Another goal: Wu's formula for $\mathrm{w}_k(TM)$.

Review:

Euler class $e(\xi) \in H^n(B;\mathbb{Z})$ is the image of the Thom class:

$$u \in H^n(E, E_0) \to H^n E \xleftarrow[\cong]{\pi^*} H^n B \ni e(\xi)$$

Submanifold $M^n \subset A^{n+k}$.

11.2: If M is closed in A, then,

$$u_\nu \in H^k(E(\nu), E(\nu)_0) \underset{T.N.T.}{\cong} H^k(N, N-M) \xleftarrow{\cong} H^k(A, A-M) \ni u_M$$

Milnor-Stasheff class u_M as u'.

Intution for u_M: $\langle u_M, [X]\rangle = M \cdot X$.

11.3: $u_M \in H^k(A, A-M) \xrightarrow{i^*} H^k A \xrightarrow{j^*} H^k M$

a) M closed in A implies $u_M \mapsto \mathrm{w}_k(\nu)$. ν oriented implies $u_M \mapsto e(\nu)$.

b) M, A closed manifolds implies $u_M \mapsto PD_A[M]$.

Application of 11.3(b): $X^k \pitchfork M^n \subset A$, all closed and oriented. In that case,

$M \cdot X = \langle u_M, [X] \rangle = \langle PD[M], X \rangle = \langle PD[M] \cup PD[X], [A] \rangle$, the algebraic intersection number.

In the case A^{n+k} closed and oriented, then, we have algebraic integral pairing:

$$\frac{H^n A}{\text{tor}} \otimes \frac{H^k A}{\text{tor}} \to \mathbb{Z}$$

$$a \otimes b \mapsto \langle a \cup b, [A] \rangle$$

Choose $\mathbb{Z}$-basis $\{e_k\}$, that gives us $\langle e_i \otimes e_j, [A] \rangle$. It's a symmetric matrix, and P.D. implies det $= \pm 1$.

Tangent Bundle

'Normal bundle of the diagonal bundle is the tangent bundle of the manifold.'

Define diagonal map $\Delta : M \to M \times M, \Delta(x) = (x, x)$.

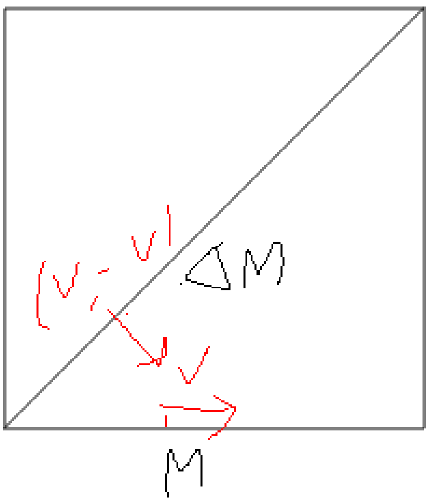


Figure 8: Diagonal Map

Consider curve $\alpha : \mathbb{R} \to M \times M$. Then we in fact have two maps: $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_i : \mathbb{R} \to M$.

Therefore, $T(M \times M) = TM \times TM$.

Notice that for any curve $\gamma : \mathbb{R} \to M$ we can find a new curve $(\gamma(t), \gamma(-t)) : \mathbb{R} \to M \times M$.

These give us lemma 11.5

Lemma 79 (11.5)**.** $\exists$ bundle map:

$$v \longmapsto (v, -v)$$

$$\begin{array}{ccc} TM & \longrightarrow & \nu(\Delta(M) \hookrightarrow M \times M) \\ \downarrow & & \downarrow \\ M & \xrightarrow[\cong]{\Delta} & M \times M \end{array}$$

Therefore $TM = \nu(\Delta \hookrightarrow M \times M)$.

Now we jump into the algebraic topology.

$$H^n(M \times M, M \times M - \Delta M) \to H^n(M \times M)$$

$$u_{\Delta M} \mapsto u''$$

Here u'' is the 'diagonal cohomology class'. $u'' = PD_{M \times M}[\Delta]$.

Lemma 80 (11.8)**.** $(1 \times a) \cup u'' = (a \times 1) \cup u''$ for $a \in H^*M$.

Sketch. $\Delta M \hookrightarrow M \times M$ is symmetric in the two factors. □

Lemma 81 (11.9)**.** When M is closed, if we take the 'slant product' then $u''/[M] = 1 \in H^0M$

Proof ommitted.

Products

Recall: Cup products $\leftrightarrow$ cross products. Implies cohomology is a ring.

Cap products imply homology is a module over cohomology ring. It corresponds to 'slant prduct'.

$$/ : H^{p+q}(X \times Y) \otimes H_qY \to H^pX$$

$$a \otimes z \mapsto a/z$$

It is supposed to be like a fraction.

It is also related to the cross product: $(a \times b)/\beta = \langle b, \beta \rangle a$.

This can work as a definition if coefficients are in a field. Theorem for general coefficients.

Definition (Slant Product)**.** At the cochain level: take $f \in H^{p+q}(X \times Y)$ and $\sigma : \Delta^q \to Y$, then for any p-chain τ,

$$(f/\sigma)(\tau) = f({}_p\sigma \times \tau)$$

[Note: this is not quite right]

Wednesday, 11/5/2025

Recap:

Slant product $/ : H^{p+q}(X \times Y) \otimes H_q X \to H^p Y$. $p \otimes \beta \mapsto p/\beta$.

Main idea: if $a \in H^p X, b \in H^q Y$ then $(a \times b)/\beta = \langle b, \beta \rangle a$.

$-/\beta$ is H^*X-linear: $((a \times 1) \cup p)/\beta = a \cup (p/\beta)$.

If M is oriented assume field coefficient F. Otherwise assume $\mathbb{F}_2$-coefficients.

Now assume that M^n is closed and smooth. $H^n(M \times M, M \times M - \Delta) \ni u_\Delta$, the thom class of the diagonal. u_Δ maps to $u'' \in H^n(M \times M)$. It is called the diagonal cohomology class, which is the Poincaré dual to ΔM.

Recall when $n \in \dim B - \dim A$, we have $H^n(B, B - A) \cong H^n(E(\nu), E(\nu)_0)$ where ν is the normal bundle by excision and tubular neighborhood theorem.

11.8: $\forall a \in H^*M, (a \times 1) \cup u'' = (1 \times a) \cup u''$, symmetry.

11.9: $u''/[M] = 1 \in H^0 M$.

Proof omitted.

11.10: Duality Theorem: $\forall$ basis $b_1, \cdots, b_r$ for H^*M there exists dual basis $b_1^\#, \cdots, b_r^\#$ so that $\langle b_i \cup b_j^\#, [M] \rangle = \delta_{ij}$.

11.11 $u'' = \sum_i (-1)^{|b_i|} b_i \times b_i^\# \in H^n(M \times M)$.

11.10 $\iff$ $I : H^*M \otimes_F H^*M \to F$ given by $a \otimes b \mapsto \langle a \cup b, [M] \rangle$ is a perfect pairing, thus $\dim H_p M = \dim H^{n-p} M = \dim H_{n-p} M$.

Suppose A, B are Λ-modules where Λ is a commutative ring. then $A \otimes_\Lambda B \to C$ is perfect pairing if $A \xrightarrow{\approx} \operatorname{Hom}(B, C)$ and $B \xrightarrow{\approx} \operatorname{Hom}(A, C)$. In our example the perfect pairing comes from the bilinear map.

Proof. We prove 11.10 and 11.11.

By Künneth theorem we can write $H^n(M \times M) \ni u'' = b_1 \times c_1 + \cdots + b_r \times c_r$.

11.8 $\implies$ $(a \times 1) \cup u'' = (1 \times a) \cup u''$. By taking slat with fundamental class,

$((a \times 1) \cup u'')/[M] = ((1 \times a) \cup u'')/[M]$

$a \cup (u''/[M]) = (1 \times a) \cup \left(\sum_j b_j \times c_j \right) /[M]$

$a = \left(\sum_j (-1)^{|a||b_j|} (1 \cup b_j) \times (a \cup c_j) \right) /[M]$.

$a = \sum_j (-1)^{|a||b_j|} \langle a \cup c_j, [M] \rangle b_j$.

Now take $a = b_i$. The b_i are a basis. Therefore, taking $a = b_i$ we see:

$$b_i = \sum_j (-1)^{|b_i||b_j|} \langle b_i \cup c_j, [M] \rangle = \delta_{ij}$$

Define $b_j^\# = (-1)^{b_j} c_j$.

$\langle b_i \cup b_j^{\#}, [M]\rangle = \delta_{ij}$

$u'' = \sum_i (-1)^{|b_i|} b_i \times b_i^{\#}$

When $M = \mathbb{R}P^2, u'' = 1 \times a^2 + a \times a + a^2 \times 1 \in H^2(\mathbb{R}^2 \times \mathbb{R}P^2)$.

□

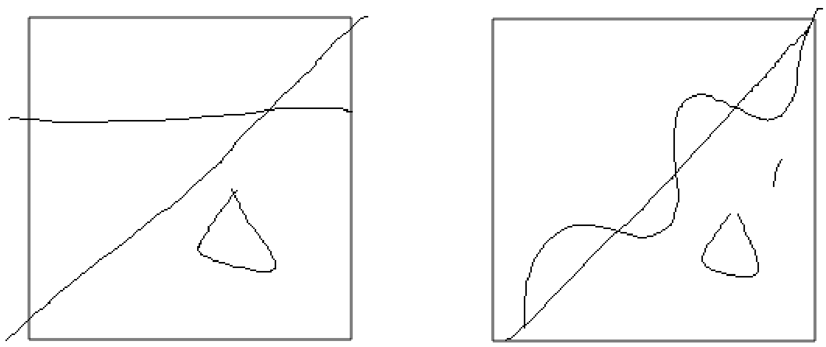
Corollary 82. When M^n is closed, smooth and oriented, $\langle e(TM), [M]\rangle = \chi(M)$.

When M^n is closed and smooth, $\langle \mathrm{w}_n(TM), [M]\rangle \cong \chi\xi(M) \mod 2$,

Proof. Oriented case: claim $e(TM) = \Delta^* u''$.

$$\begin{tikzcd}
 u_{TM} & & e(TM) \\
 H^n(TM, TM_0) \arrow[rr] \arrow[d, "\cong"] & & H^n(M \times M) \\
 H^n(E(\nu : \Delta \hookrightarrow M \times M), E(v)_0) \arrow[r, "\cong"] & H^n(M \times M, M \times M - \Delta) \arrow[r] & H^*(M \times M) \arrow[u, "\Delta^*"'] \\
 & u_\Delta & u''
\end{tikzcd}$$

□



Let Δ' be isotopic copy of Δ such that $\Delta' \pitchfork \Delta$.

Then $\Delta' \cdot \Delta = \langle e(\nu), [\Delta]\rangle - \langle e(TM), [M]\rangle = \chi(M)$

THus, $\chi(M)$ is the self intersection number of the diagonal $\Delta M \hookrightarrow M \times M$

Corollary 83. If M has a nowhere zero vector field then $\chi(M) = 0$.

Proof. Suppose otherwise. Then M has a non-zero vector field implies ΔM has a non-zero normal vector field. "Flow" implies $\exists \Delta'$ such that $\Delta' \cap \Delta = \varnothing$. □

Thus, $\chi(M) \neq 0 \implies$ can't comb hairy M.

Recall $\chi(M) = (-1)^i \dim H_i(M, \mathbb{Q}) = \sum_i (-1)^i$(#-of i-cells).

$= \sum_i (-1)^i \dim H_i(M, \mathbb{F}_p)$.

Friday, 11/7/2025

Wu classes / Wu Formula / Wu Theorem

Coefficients in $\mathbb{F}_2$ understood.

Wu classes are polynomials of whitney classes.

$\mathrm{v}_0 = \mathrm{w}_0 = 1$

$\mathrm{v}_1 = \mathrm{w}_1$

$\mathrm{v}_2 = \mathrm{w}_1^2 + \mathrm{w}_2$

$\mathrm{v}_3 = \mathrm{w}_1 \mathrm{w}_2$.

They're defined as following:

Definition (Total Wu Class)**.**

$$\mathrm{v} = \mathrm{v}_0 + \mathrm{v}_1 + \mathrm{v}_2 + \cdots$$

$$\mathrm{w} = \mathrm{Sq}\, v$$

i.e. $v = \mathrm{Sq}^{-1} \mathrm{w} = (1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots)^{-1} \mathrm{w}$.

Proposition 84 (Wu's Formula, Execrise 8A)**.** $\mathrm{Sq}^k \mathrm{w}_m$ is 'something in the cohomology of the Grassmanian', so it must be some polynomial over Stiefel Whitney Classes.

$$\mathrm{Sq}^k \mathrm{w}_m = \sum_i \binom{k-m}{i} \mathrm{w}_{k-i} \mathrm{w}_{m+i}$$

Hint on 8A:

$$H^*(G_n) \cong H^*(P^m \times \cdots \times P^m)^{S_n}$$

$$\mathrm{w}_i \mapsto \sigma_i(a_1, \cdots, a_n)$$

Compute Sq^i using Cartan.

eg $\mathrm{Sq}^1 \mathrm{w}_2 = \mathrm{w}_1 \mathrm{w}_2 + \mathrm{w}_3$.

Computation:

$$\mathrm{w} = \mathrm{Sq}\,\mathrm{v} = (1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots)(v_0 + v_1 + v_2 + \cdots)$$

Then, $1 = \mathrm{w}_0 = \mathrm{v}_0$.

$\mathrm{w}_1 = \mathrm{v}_1$

$\mathrm{w}_2 = \mathrm{Sq}^1 \mathrm{v}_1 + \mathrm{v}_2 \implies \mathrm{w}_2 = \mathrm{w}_1^2 + \mathrm{v}_2$

$\mathrm{w}_3 = \cancel{\mathrm{Sq}^2 \mathrm{v}_1} + \mathrm{Sq}^1 \mathrm{v}_2 + \mathrm{v}_3 = \mathrm{Sq}^1 \mathrm{w}_1^2 + \mathrm{Sq}^1 \mathrm{w}_2 + \mathrm{v}_3$

$= \underbrace{\cancel{\mathrm{Sq}^0 \mathrm{w}_1 \mathrm{Sq}^1 \mathrm{w}_1 + \mathrm{Sq}^1 \mathrm{w}_1 \mathrm{Sq}^0 \mathrm{w}_1}}_{\text{cartan}} + \underbrace{\mathrm{w}_1 \mathrm{w}_2 + \mathrm{w}_3}_{\text{Wu Formula}} + \mathrm{v}_3$

Now, suppose we have M^n a closed n-manifold.

Theorem 85 (Wu Theorem)**.** Let $\mathrm{v}(TM)$ be the total Wu class of a tangent bundle.

$$\langle \mathrm{v}(TM) \cup -, [M] \rangle = \langle \mathrm{Sq}(-), [M] \rangle$$

i.e. if $x \in H^{n-k}M$ then $\mathrm{v}_k(TM) \cup x = \mathrm{Sq}^k x$.

i.e. $\langle \mathrm{v}_k(TM) \cup x, [M] \rangle = \langle \mathrm{Sq}(-), [M] \rangle$.

Corollary 86. Let $M \underset{h}{\simeq} M'$ be homotopy equivalent manifolds. Then, $\mathrm{w}(TM) = h^* \mathrm{w}(TM')$.

Sketch. Wu classes are determined by algebraic topology. Thus, homotopy equivalent implies same algebraic topology which implies same Wu class which implies same Stiefel-Whitney class. ☐

We can connect this to intersection forms.

Definition (Algebraic Intersection Form)**.** $I_M : H^*M \otimes H^*M \to \mathbb{F}_2$.

$$I_M(a \otimes b) = \langle a \cup b, [M] \rangle$$

We write $a \cdot b = I_M(a \otimes b)$. By Poincaré duality it is a *perfect pairing*, thus it is a *non-singular pairing.*

Key application of Wu's Theorem

Suppose $n = 2k$. M is a closed n-dimensional manifold.

$$\langle v_k(TM) \cup x, [M] \rangle = \langle \mathrm{Sk}^k x, [M] \rangle = \langle x \cup x, [M] \rangle$$

Thus, for $x \in H^kM$:

$$\mathrm{v}_k(TM) \cdot x = x \cdot x\ .$$

Now we restrict to the middle dimensional homology.

$$\widehat{I_M} : H^kM^{2k} \otimes H^kM^{2k} \to \mathbb{F}_2$$

Definition. $\widehat{I_M}$ is even if $\forall a, \widehat{I_M}(a \otimes a) = 0$.

$\iff$ if β_i is a basis for $H^k M$ then the matrix $(\beta_i \cdot \beta_j)$ has even # on the diagonal.

Then,

Theorem 87 (Wu's Theorem)**.**

$$\mathrm{v}_k(TM^{2k}) = 0 \iff \widehat{I_M} \text{ is even}$$

Example: Suppose $n = 2$. Then $\mathrm{v}_1 = \mathrm{w}_1$.

$\mathrm{v}_1 = 0 \iff M^2$ orientable $\iff \widehat{I_M}$ is even (eg Torus).

Matrix: $\begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$

$\mathrm{v}_1 \neq 0 \iff \widehat{I_M}$ is odd. e.g. $\mathbb{R}P^1 \cdot \mathbb{R}P^1 = 1$ in $\mathbb{R}P^2$.

Let K be the Klein bottle. Then $\widehat{I_K}$ has matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ since $b \cdot b = 1$ and $a \cdot a = 0$.

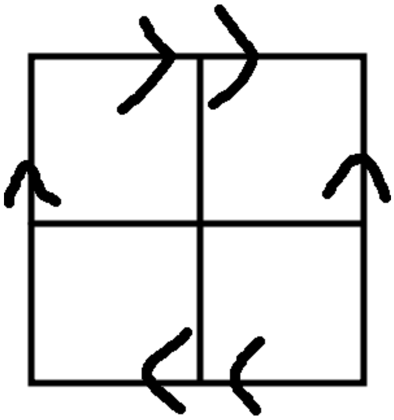


Figure 9: Klein Bottle

e.g. $\mathbb{R}P^4 : \mathrm{w}_1 \neq 0, \mathrm{w}_2 = 0, \mathrm{v}_2 \neq 0$ thus $\mathbb{R}P^2 \cdot \mathbb{R}P^2 = 1$.

Further example: $\mathbb{C}P^1 \cdot \mathbb{C}P^1 = 1$.

Corollary 88. Orientable 4-manifold: $\widehat{I_M}$ is even $\iff \mathrm{w}_2(TM) = 0 \iff \mathrm{v}_2(TM) = 0$.

To prove Wu's theorem we need an additional lemma:

Lemma 89 (11.3)**.**

$$\mathrm{w}(TM) = \mathrm{Sq}(u'')/[M]$$

Where $u'' \in H^n(M \times M)$ the diagonal cohomology class dual to ΔM.

Proof. We assume the lemma is true. In that case,

I_M is perfdect pairingm thus non-singular, thus $\exists! \hat{v} \in H^*M$ such that $\langle \hat{v} \cup -, [M] \rangle = \langle \mathrm{Sq}(-), [M] \rangle : H^*M \to \mathbb{F}_2$.

WTS: $\hat{v} = \mathrm{v}(TM)$.

WTS: $\mathrm{Sq}\, \hat{v} = \mathrm{w}(TM)$.

Choose basis b_i for H^*M and dual basis $b_i^\sharp$ i.e. $b_i \cdot b_j^\sharp = \delta_{ij}$ [11.10]

Then, 11.11 $\implies u'' = \sum_i b_i \times b_i^\sharp$.

11.10: $\hat{v} = \left(\sum_i \hat{v} \cdot b_i^\sharp \right) b_i = \sum_i \langle \mathrm{Sq}(b_i^\sharp), [M] \rangle b_i$

$\implies \mathrm{Sq}\, \hat{v} = \sum_i \langle \mathrm{Sq}(b_i^\sharp), [M] \rangle \,\mathrm{Sq}\, b_i$

Cartan and 11.11 implies,

$\mathrm{Sq}\, \hat{v} = \sum_i (\mathrm{Sq}(b_i) \times \mathrm{Sq}(b_i^\sharp))/[M] = \mathrm{Sq}(u'')/[M] = \mathrm{w}(TM)$.

□

Monday, 11/10/2025

Recap: Wu classes: $\mathrm{Sq}\, v = \mathrm{w}$.

Wu formula:

$$\mathrm{Sq}^k \mathrm{w}_m = \sum_i \binom{k-m}{i} \mathrm{w}_{k-i} \mathrm{w}_{k+i}$$

Using these, we can find out: $v_1 = \mathrm{w}_1, v_2 = \mathrm{w}_1^2 + \mathrm{w}_2, v_3 = \mathrm{w}_1 \mathrm{w}_2$.

Wu's Theorem: If M is a closed manifold and $x \in H^*(M; \mathbb{F}_2)$ then,

$$\langle v(TM) \cup x, [M] \rangle = \langle \mathrm{Sq}^k(x), [M] \rangle$$

Corollary 90. If $k > \frac{\dim M}{2}$ then $v_k(TM) = 0$.

Proof. $\forall x \in H^{n-k}(M; \mathbb{F}_2)$

$$\langle v_k(TM) \cup x, [M] \rangle = \langle \mathrm{Sq}^k(x), [M] \rangle = \langle 0, [M] \rangle = 0$$

□

If $k = \frac{\dim M}{2}$ then $\langle v_k(TM) \cup x, [M] \rangle = \langle x \cup x, [M] \rangle$ which is the 'self intersection' number.

Application to 3-manifolds

Let M^3 be closed, $\mathrm{w}_i = \mathrm{w}_i(M), v_i = v_i(TM)$.

Theorem 91. a) All SW numbers of M^3 vanish.

b) M^3 orienatable implies $\mathrm{w}_1 = \mathrm{w}_2 = \mathrm{w}_3 = 0$.

Proof. $\frac{\dim M}{2} \implies v_2 = 0, v_3 = 0$. Then $\mathrm{w}_1^2 = \mathrm{w}_2$ and $\mathrm{w}_1 \mathrm{w}_2 = 0$. So $\mathrm{w}_1^3 = 0$. $\chi(M^3) = 0 \implies \mathrm{w}_3 = 0$ [recall $\chi(M^n) \equiv \langle \mathrm{w}_n(TM), [M] \rangle \pmod 2$, apply PD].

For part b: $\mathrm{w}_1 = 0 \implies \mathrm{w}_2 = 0, \mathrm{w}_3 = 0$. ◻

a + Thom's theorem $\implies M^3 = \partial W^4$ compact, i.e. every 3-manifold is the boundary of a compact 4-manifold.

b + obstruction theorem $\implies$ oriented closed 3-manifold M^3 has trivial tangent bundle, "paralellizable" [Problem 12-13].

Gysin Sequence

It's a long exact sequence. Consider the vector bundle
$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}.$$

a) $\exists$ LES:

$$\cdots \to H^{j-n}(B; \mathbb{F}_2) \xrightarrow{-\cup \mathrm{w}_n} H^j(B; \mathbb{F}_2) \xrightarrow{\pi^\sharp} H^j(E_0; \mathbb{F}_2) \to H^{j-n+1}(B; \mathbb{F}_2) \to \cdots$$

b) If oriented, $\exists$ LES:

$$\cdots \to H^{j-n} \xrightarrow{-\cup e} H^j B \to H^j E_0 \to H^{j-n+1} B \to \cdots$$

c) If oriented with metric,

$$\cdots \to H^{j-n} B \xrightarrow{-\cup e} H^j B \to H^j(S(E)) \to \cdots$$

Recall, suppose we have a trivial bundle. $H^* E_0 = H^*(B \times (\mathbb{R}^n - 0)) = H^*(B \times S^{n-1}) = H^* B \oplus H^{*+n-1} B$ [Künneth]. Since in trivial bundle, $- \cup e$ is 0 this works!

Proof. b: LES of pair (E, E_0):

$$\begin{tikzcd} H^j(E, E_0) \arrow[r] & H^j E \arrow[r] & H^j E_0 \arrow[r] & H^{j+1}(E, E_0) \\ H^{j-n} \arrow[u, "\cong", "-\cup u"'] \arrow[ur, "-\cup u|_E"] & H^j B \arrow[u, "\cong"] & & \\ H^{j-n} B \arrow[u, "\cong"] \arrow[ur, "-\cup e"] & & & \end{tikzcd}$$

2nd proof: SSS tp
$$\begin{tikzcd} S^1 \arrow[r] & E \arrow[d] \\ & S^2 \end{tikzcd}.$$

Classified by $e \in H^2(S^2)$. eg $E = S^3, S^1 \times S^2, L_n$ lens spaces, $e = 0, 1, n$. ◻

Corollary 92 (12.3)**.** Any 2-fold cover $\begin{array}{c}\widetilde{B}\\ \big\downarrow \pi\\ B\end{array}$ implies: $\exists \xi = \begin{array}{ccc}\mathbb{R} & \longrightarrow & E\\ & & \big\downarrow\\ & & B\end{array}$ such that,

$$\begin{tikzcd} SE \arrow[rr, "\cong"] \arrow[dr] & & \widetilde{B} \arrow[dl] \\ & B & \end{tikzcd}$$

and LES:

$$\cdots \to H^{j-1}(B;\mathbb{F}_2) \xrightarrow{-\cup \mathrm{w}_1} H^j(B;\mathbb{F}_2) \to H^j(\widetilde{B},\mathbb{F}_2) \to \cdots$$

'Smith exact sequence, Hatcher'

Proof. Let $E \coloneqq \frac{\widetilde{B}\times\mathbb{R}}{(x,t)\sim(x',-t)}$

Where $\pi(x) = \pi(x'), x \neq x'$.

Use Gysin. e.g. $\begin{array}{c}S^2\\ \big\downarrow\\ P^2\end{array}$ or $\begin{array}{c}T^2\\ \big\downarrow\\ K^2\end{array}$. ∎

$\widetilde{G}_n(\mathbb{R}^{n+k})$ = oriented n-planes in $\mathbb{R}^{n+k}$. This is $V_n(\mathbb{R}^{n+k})/SO(n)$.

$$V_n(\mathbb{R}^{n+k}) = \{(v_1,\cdots,v_n) \mid v_i \in \mathbb{R}^{n+k}; v_i\cdot v_j = \delta_{ij}\} \subset \mathbb{R}^{n+k}\times\cdots\times\mathbb{R}^{n+k}$$

$SO(n) = \{A \in M_n\mathbb{R} \mid AA^t = I, \det A = 1\}$.

$$\begin{array}{ccccc}\widetilde{G}_n & = & \widetilde{G}_n(\mathbb{R}^\infty) & = & BSO(n)\\ \big\downarrow{\scriptstyle \text{double cover}} & & & & \\ G_n & = & G_n(\mathbb{R}^\infty) & = & BO(n)\end{array}$$

Then we will have 12.3 (Gysin):

$$H^*(\widetilde{G}_n;\mathbb{F}_2) = \mathbb{F}_2[\mathrm{w}_2,\mathrm{w}_3,\cdots]$$

Friday, 11/14/2025

Today: $\widetilde{G_n}$ and $\mathbb{C}$ vector bundles.

Definition (Oriented Grassmanian)**.** $\widetilde{G_n}(\mathbb{R}^{n+k})$ = oriented n-planes in $\mathbb{R}^{n+k}$

$$= \frac{\text{Orthonormal } n\text{-frames in } \mathbb{R}^{n+k}}{\text{Orientation Preserving Rigit motions}} = \frac{V_n(\mathbb{R}^{n+k})}{\mathrm{SO}(n)}$$

Then there's a double cover:

$$\begin{array}{c} \widetilde{G}_n(\mathbb{R}^{n+k}) \\ \big\downarrow \\ G_n(\mathbb{R}^{n+k}) \end{array}$$

The double cover is not trivial. $k > 0, \widetilde{G}_n(\mathbb{R}^{n+k})$ is connected.

There is a tautological bundle over this space.

$$\begin{array}{c} E(\widetilde{\gamma}_n) \\ \big| \\ \widetilde{G}_n(\mathbb{R}^{n+k}) \end{array}$$

Definition 1: $E(\widetilde{\gamma}_n) \subset \widetilde{G}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$.

Definition 2: Pullback:

$$\begin{tikzcd} E(\widetilde{\gamma}_n) \arrow[r] \arrow[d] & E(\gamma_n) \arrow[d] \\ \widetilde{G}_n(\mathbb{R}^{n+k}) \arrow[r] & G_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

$\widetilde{G}_n = G_n(\mathbb{R}^\infty) = \operatorname*{colim}_{k\to\infty} \widetilde{G}_n(\mathbb{R}^{n+k})$

Theorem 93. $\begin{array}{c} E(\widetilde{\gamma}_n) \\ \big\downarrow \\ \widetilde{G}_n \end{array}$ classifies oriented vector bundles over B CW. i.e.

$$[B, \widetilde{G}_n] \longleftrightarrow \left\{ \begin{array}{c} \text{iso class of} \\ \text{oriented } n\text{-planes} \\ \text{bundles } /B \end{array} \right\}$$

$$f \longmapsto f^*\widetilde{\gamma}_n$$

$H^*(\widetilde{G}_n) \to H^*B$. Here $\widetilde{G}_n$ classifying space, $\widetilde{\gamma}_n$ universal bundle.

Proof. First: If ξ oriented then any bundle map $\xi \to \gamma_n$ lifts uniquely to o.p. bundle map $\xi \to \widetilde{\gamma}_n$

Second: Presentation

$$\begin{tikzcd} SO(n) \arrow[r] & V_n(\mathbb{R}^\infty) \arrow[d] & \simeq & * \\ & \widetilde{G}_n & & \end{tikzcd}$$

Then $\widetilde{G}_n = BSO(n) = BGL_n^+(\mathbb{R})$. $\square$

$\widetilde{G}_n \xrightarrow{\pi} G_n$: non-trivial 2-fold cover. Let γ_π be the associated line bundle to the double cover.

$H^1(G_n, \mathbb{F}_2) = \mathbb{F}_2$. This is w_1.

Therefore, $\mathrm{w}_1(\gamma_\pi) = \mathrm{w}_1(\gamma_n)$.

We can change the fiber:

$$\begin{tikzcd}
 & & & \widetilde{G}_n \times_{C_2} \mathbb{R} \arrow[d, equal] \\
S^0 \arrow[r] & \widetilde{G}_n \arrow[d] & \mathbb{R} \arrow[r] & E(\gamma_\pi) \arrow[d] \\
 & G_n & & G_n
\end{tikzcd}$$

Recall 12.3: Gysin sequence for γ_π.

$$\xrightarrow{0} H^{j-1}(G_n; \mathbb{F}_2) \xrightarrow{-\cup \mathrm{w}_1} H^j(G_n; \mathbb{F}_2) \to H^j(\widetilde{G}_n; \mathbb{F}_2) \xrightarrow{0} H^j(G_n; \mathbb{F}_2) \xrightarrow{-\cup \mathrm{w}_1}$$

$H^*(G_n; \mathbb{F}_2) = \mathbb{F}_2[\mathrm{w}_1, \cdots, \mathrm{w}_n]$. This is a polynomial ring, so multiplying by w_1 is injective.

Thus, $- \cup \mathrm{w}_1$ is injective.

Theorem 94 (12.4)**.** $H^*(\widetilde{G}_n; \mathbb{F}_2)/\langle \mathrm{w}_1 \rangle = \mathbb{F}_2[\mathrm{w}_2, \cdots, \mathrm{w}_3]$

Remark: there also exists Euler class $e(\widetilde{\gamma}_n) \in H^n(\widetilde{G}_n; \mathbb{Z})$

If we have an oriented v.b. ξ, then $e(\xi) \in H^n(B; \mathbb{Z})$. n odd means $2e(\xi) = 0$.

Q(Davis): Find example where $e(\xi) \neq 0, n$ odd.

A(Mandell): $\xi = \widetilde{\gamma}_3$, oriented grassmanian of 3-planes in $\mathbb{R}^\infty$.

$e(\widetilde{\gamma}_3) \in H^3(\widetilde{G}_3; \mathbb{Z})$

$0 \neq e(\widetilde{\gamma}_3) \xrightarrow{mod} \mathrm{w}_3 = \mathrm{w}_3(\widetilde{\gamma}_3) \neq 0$.

Puzzles:

1. What 2-dimensional real planes in $\mathbb{C}^n$ are complex lines?

2. P176:

$$\begin{tikzcd}
 & \text{real} \arrow[dr, bend left] & \\
\text{oriented} \arrow[ur, bend left] & & \text{complex} \arrow[ll, bend left]
\end{tikzcd}$$

$\mathbb{C}^n$-bundle

$$
w \qquad \begin{tikzcd} \mathbb{C}^n \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}
$$

MS Definition, of Steenrod $\mathrm{GL}_n(\mathbb{C}, \mathbb{C}^n)$-bundle

Complex projective space $\mathbb{C}P^n = G_1(\mathbb{C}^{n+1})$.

Complex Grassmanian $G_n(\mathbb{C}^{n+k})$

Tautological bundle:

$$
\begin{tikzcd} \mathbb{C}^n \arrow[r] & E(\gamma_n) \arrow[d] & \subset & G_n(\mathbb{C}^{n+k}) \times \mathbb{C}^{n+k} \\ & G_n(\mathbb{C}^{n+k}) \end{tikzcd}
$$

Universal bundle:

$$
\begin{tikzcd} \mathbb{C}^n \arrow[r] & E(\gamma^n) \arrow[d] \\ & G_n(\mathbb{C}^\infty) \end{tikzcd}
$$

$H^*(G_n\mathbb{C}^\infty)$ characterstic classes, $\mathbb{C}^n$-bundle.

$H^*(G_n\mathbb{C}^\infty, \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \cdots, c_n]$ are called Chern Classes.

$|c_i| = 2i$.

$$
\mathbb{C}^n\text{-bundle} \longrightarrow \mathbb{R}^{2n}\text{-bundle}
$$

$$
w \longmapsto w\big|_{\mathbb{R}}
$$

Definition. A complex structure on $\xi : \mathbb{R}^n \to E \to B$ is a bundle map:

$$
\begin{tikzcd} E(\xi) \arrow[rr, "J"] \arrow[dr] & & E(\xi) \arrow[dl] \\ & B & \end{tikzcd}
$$

such that $J^2 = -\,\mathrm{id}$. i.e. $J(J(v)) = -v$.

$$
\text{complex vector bundle} \longleftrightarrow \text{real vector bundle with complex structure}
$$

Monday, 11/17/2025

1. $\mathrm{Spin}(n) \to \mathrm{SO}(n)$

2. BoH Periodicity:

$$\pi_i O = \begin{cases} \mathbb{Z}/2, & \text{if } i \equiv 0(8); \\ \mathbb{Z}/2, & \text{if } i \equiv 1(8); \\ 0, & \text{if } i \equiv 2(8); \\ \mathbb{Z}, & \text{if } i \equiv 3(8); \\ 0, & \text{if } i \equiv 4(8); \\ 0, & \text{if } i \equiv 5(8); \\ 0, & \text{if } i \equiv 6(8); \\ \mathbb{Z}, & \text{if } i \equiv 7(8); \end{cases}$$

$\pi_1 U = \begin{cases} 0, & \text{if } i \equiv 0(2); \\ \mathbb{Z}, & \text{if } i \equiv 1(2). \end{cases}$

3. Splitting principal

$$\begin{tikzcd} L_1 \oplus \cdots \oplus L_n \arrow[r] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

f^* injective.

Homotopy

$\pi_i(X, x_0) = [(S^i, *), (X, x_0)]$.

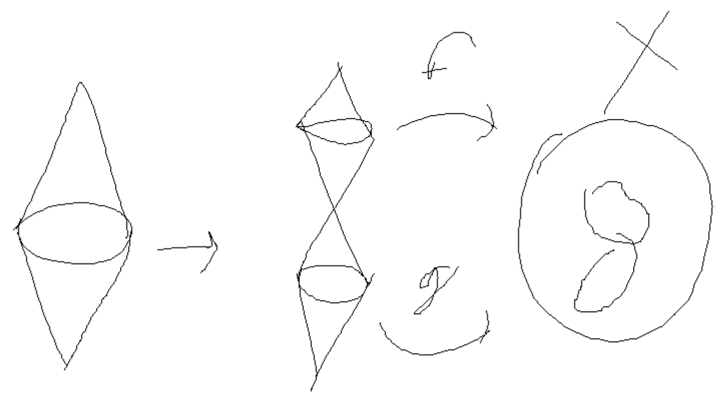
$i = 0 : \pi_0 \leftrightarrow$ path-component of X.

$i \geq 2$: Abelian group.

Suppose X is path connected.

Path $\gamma : I \to X$ with $\gamma_* : \pi_i(X, \gamma(0)) \xrightarrow{\approx} \pi_i(X, \gamma(1))$. So we can omit x_0 from the definition. We can go wrong sometimes, but we won't worry about it.

Addition structure:



$\pi_i GL_n(\mathbb{R}) = \mathrm{Vect}_n(S^{i+1})$ isomorphisom classes.

$\mathrm{Vect}_n(S^{i+1})$ is
$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & E \arrow[d] \\ & S^{i+1} \end{tikzcd}$$

Proof 1. Clutching.

$\xi\big|_{H_+^{i+1}}$ and $\xi\big|_{H_-^{i+1}}$ are trivial. ξ is given, $S^i \to GL_n(\mathbb{R})$ by gluing. □

Proof 2.
$$\begin{tikzcd} GL_n \arrow[r] & EGL_n \arrow[d] & \simeq & * \\ & BGL_n & & \end{tikzcd}$$

Then $\mathrm{Vect}_n(S^{i+1}) \overset{C.S.}{=} \pi_{i+1} BGL_n \underset{LES}{\cong} \pi_i GL_n$ □

In general $[X, BG] \cong$ Iso class of (G, F)-bundle $/X$.

Classifying Spaces

We have the following groups:

$$\begin{tikzcd} \mathrm{SO}(n) \arrow[r, hook] \arrow[d, hook] & \mathrm{GL}_n^+(\mathbb{R}) \arrow[d, hook] \\ O(n) \arrow[r, hook] & \mathrm{GL}_n(\mathbb{R}) \end{tikzcd}$$

$\mathrm{GL}_n^+(\mathbb{R})$ corresponds to orientable bundles.

$\mathrm{O}(n)$ corresponds to metrics.

Claim: the horizontal maps are homotopy equivalent

Proof. Polar decomposition: $A \in \mathrm{GL}_n(\mathbb{R}) \implies A = PO$ where P is 'positive' [i.e. symmetric and positive definite] and $O \in \mathrm{O}(n)$.

Then $\mathrm{O}(n)$ is a deformation retract of GL_n by

$$((1-t)P + tI)O$$

$\square$

Corollary 95. $\mathrm{BO}(n) \simeq \mathrm{BGL}_n \mathbb{R}$

Every bundle over CW-complex admit a metric / unique upto isometry.

Theorem 96. $\mathrm{SO}(n)$ is path-connected, $\pi_0 \mathrm{O}(n) \xrightarrow{\approx} [\det]\{\pm 1\}$.

Proof. Pick $0 \neq a \in \mathbb{R}^n$. Look at reflection through $a^\perp$. Call it R_a.

Then $R_a : \mathbb{R}^n \to \mathbb{R}^n$, $R_a\big|_{a^\perp} = \mathrm{id}, R_a(a) = -a$.

First, if $O \in \mathrm{O}(n)$ then O is a product of reflection.

Second, if $S \in \mathrm{SO}(n)$ then S is a product of even number of reflection.

Third, if a, b are linearly independent then $R_a \simeq R_b$ via $R_{ta+(1-t)b}$.

Fourth, $R_a R_b \simeq R_a R_a = \mathrm{id}$.

This proves the problem. Note that $AA^t = 1 \implies (\det A)^2 = 1 \implies \det A \in \{\pm 1\}$. $\square$

Then $\mathrm{SO}(n)$ is path-connected and $\mathrm{O}(n)$ has two path components.

Wednesday, 11/19/2025

Let $R = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$

Then we have the following split exact sequence:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SO}(n) \arrow[r] & \mathrm{O}(n) \arrow[r, "\det"] & \{\pm 1\} \arrow[r] \arrow[l, bend left, dotted] & 1 \end{tikzcd}$$

$$R \longleftarrow\!\shortmid -1$$

Then $\mathrm{O}(n) = \mathrm{SO}(n) \rtimes \{\pm 1\}$.

$\pi_0 \mathrm{O}(n) = \{\pm 1\}$.

$\mathrm{SO}(1) = \{1\}, \mathrm{O}(1) = \{\pm 1\}$.

$\mathrm{SO}(2) = S^1, \mathrm{O}(2) = S^1 \rtimes \{\pm 1\}$ the dihedral group.

Lemma 97. $\mathrm{SO}(3) \cong \mathbb{R}P^3$.

Proof 1. $A \in \mathrm{SO}(3)$. Then the characteristic polynomial is of degree 3. Thus, A has a real eigenvalue.

Since $A \in \mathrm{SO}(3)$ the eigenvalue $\lambda = \pm 1$.

Case 1: all eigenvalues are real. $\begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \notin \mathrm{SO}(3)$.

Case 2: Other eigenvalues are non-real. Then $\lambda = \pm 1, \mu, \overline{\mu}$ with $\lambda\mu\overline{\mu} = 1 \implies \lambda = 1$.

Thus, there exists 'axis' v such that $Av = v$ with $\overline{v} = 1$.

i.e. A is a rotation about axis v through angle $0 \leq \theta \leq \pi$.

$$\mathrm{SO}(3) \xrightarrow{\approx} D^3/\sim = \mathbb{R}P^3$$

$$A \mapsto \frac{\theta}{\pi} v$$

□

Proof 2. $S^3 =$ unit quarternions $= \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\}$.

Claim: S^3 is a double cover of $\mathrm{SO}(3)$. We essentially have to prove that:

$$\begin{tikzcd} 1 \arrow[r] & \{\pm 1\} \arrow[r] & S^3 \arrow[rr] & & \mathrm{SO}(3) \arrow[rr] & & 1 \end{tikzcd}$$

$$z \longmapsto (bi + cj + dk \mapsto z(bi + cj + dk)\overline{z})$$

□

Lemma 98 (Stability Lemma)**.** Recall $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1) \hookrightarrow \cdots$ by $A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$.

a) $\pi_{n-1}\,\mathrm{SO}(n) \twoheadrightarrow \pi_{n-1}\,\mathrm{SO}(n+1) \xrightarrow{\approx} \pi_{n-1}\,\mathrm{SO}(n+2) \xrightarrow{\approx}$

b) $\pi_n\,\mathrm{BSO}(n) \twoheadrightarrow \pi_n\,\mathrm{BSO}(n+1) \xrightarrow{\approx} \pi_r\,\mathrm{BSO}(n+2) \xrightarrow{\approx}$

Proof. Fiber bundle.

$$\begin{tikzcd} \mathrm{SO}(n) \arrow[r] & \mathrm{SO}(n+1) \arrow[d] & A \arrow[d, mapsto] \\ & S^n & A\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \end{tikzcd}$$

LES on π_* and $\pi_1 S^n = 0$ for $i < n$.

a) LES on π_* and $\pi_1 S^n = 0$ for $i < n$.

b)

$$\begin{tikzcd} \mathrm{SO}(n) \arrow[r] & \mathrm{ESO}(n) \arrow[d] & \simeq & * \\ & \mathrm{BSO}(n) & & \end{tikzcd}$$

$\pi_i \,\mathrm{BSO}(n) \xrightarrow{\approx} \pi_{i-1}\,\mathrm{SO}(n)$.

$\square$

Example:

$$\pi_1\,\mathrm{SO}(2) \longrightarrow \pi_1\,\mathrm{SO}(3) \xrightarrow{\approx} \pi_1\,\mathrm{SO}(4) \longrightarrow$$

$$TS^2 \longmapsto 0$$

$\pi_1\,\mathrm{SO(n)} = \mathbb{Z}_2$ for $n > 2$. $\pi_1\,\mathrm{SO}(2) = \mathbb{Z}$.

We define $\mathrm{Spin}(n)$ as connected double group of $\mathrm{SO}(n)$.

$\mathrm{Spin}(3) = S^3$.

$$0 \longrightarrow \{\pm 1\} \xrightarrow{\Delta} S^3 \times S^3 \to \mathrm{SO}(4) \longrightarrow 1$$

$$(z, w) \longmapsto (v \mapsto zw\overline{v})$$

$\mathrm{Spin}(4) = S^3 \times S^3$.

Spin structure on $\xi = \mathbb{R}^n \to E \to B$ or vector bundle with metrics where B is path-connected.

$P_{\mathrm{SO}} = \{(e_1, \cdots, e_n) \mid \pi(e_i) = \pi(e_i) = \pi(e_j),\ \text{orthonormal}\}$

Then we can define spin structure to $\mathrm{Spin}\, n$. i.e.

principal $\mathrm{Spin}(n)$:

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r] & P_{\mathrm{Spin}(n)} \arrow[d] \\ & B \end{tikzcd}$$

Furthermore,

$$\begin{tikzcd} P_{Spin} \times_{Spin} \mathrm{SO} \arrow[rr, "\approx"] \arrow[dr] & & \mathrm{P_{SO}} \arrow[dl] \\ & B & \end{tikzcd}$$

$\iff$ the following happens:

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & P_{\mathrm{SO}} \arrow[dl] \\ & B & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Spin} \arrow[rr, dash] \arrow[d] & & \mathrm{SO} \arrow[d] \\ P_{\mathrm{spin}} \arrow[rr] \arrow[dr] & & P_{\mathrm{SO}} \arrow[dl] \\ & B & \end{tikzcd}$$

Deine: $\mathrm{Spin}(n)$ as conected double cover of $\mathrm{Spin}(n)$

Theorem 99. ξ admits a spin structure $\iff$ $\mathrm{w}_2\,\xi = 0$.

If ξ admits a spin structure then,

$$\text{spin structures} \leftrightarrow H^1(B;\mathbb{Z}_2)$$

Proof.

$$\begin{tikzcd} \mathbb{R}P^\infty \arrow[r] & \mathrm{BSpin}(n) \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & \mathrm{BSO}(n) \end{tikzcd}$$

□

Monday, 12/1/2025

Let $\xi = \mathbb{R}^n \to E \to B$ be oriented with metric.

Theorem 100. ξ admits a spin structure iff $\mathrm{w}_2(\xi) = 0$.

If so, spin structure on $\xi \leftrightarrow H^1(B;\mathbb{Z}_2)$.

Consider the *frame bundle.*

$$\begin{tikzcd} \mathrm{SO}(n) = \mathrm{SO} \arrow[r] & P_{\mathrm{SO}} \arrow[d, "p"] & = & \{(e_1, \cdots, e_n) \mid p(e_i) = p(e_j), e_i \text{ O.N.}\} \subset E \times \cdots \times E \\ & B & & \end{tikzcd}$$

spin structure on $\xi \leftrightarrow \alpha \in H^1(P_{\mathrm{SO}};\mathbb{Z}_2)$ such that $i^*\alpha \neq 0$.

$\leftrightarrow \alpha : \pi_1 P_{\mathrm{SO}} \to \mathbb{Z}_2$ such that $\alpha \circ i \neq 0$.

This gives rise to the double cover $\begin{tikzcd} P_{\text{spin}} \arrow[d] \\ P_{\text{SO}} \end{tikzcd}$

Given the fibration, we have the Serre 5-term exact sequence [with $\mathbb{Z}_2$-coefficients]

$$\begin{tikzcd} H^1 B \arrow[r] & H^1 P_{\text{SO}} \arrow[r] & H^1 \operatorname{SO} \arrow[r, "\delta_3"] \arrow[d, phantom, "\parallel"] & H^2 B \arrow[r] & H^2 P_{\text{SO}} \\ & & \{0, g\} & & \end{tikzcd}$$

This is a consequence of the Serre Spetral Sequence.

Claim: $\delta_3(g) = \mathrm{w}_2(\xi)$.

Proof: (i): $\delta_3(g) \in H^1 B$ is a characteristic class for oriented vector bundle with metric [everything natural, we have a pullback].

(ii): 'universal case': $\begin{tikzcd} \operatorname{SO}(n) \arrow[r] & \operatorname{ESO}(n) \arrow[d] & \simeq & * \\ & \operatorname{BSO}(n) & = & \widehat{G}_n \end{tikzcd}$

$$\begin{tikzcd} 0 \arrow[r] & H^1(\operatorname{SO}) \arrow[r, tail, "\approx"] & H^2(\operatorname{BSO}(n)) \\ & (0, g) & (0, \mathrm{w}_2) \end{tikzcd}$$

END OF SPIN!

Recall stability lemma:

$$\begin{tikzcd} \pi_k \operatorname{O}(k+1) \arrow[r, two heads] \arrow[d, phantom, "\parallel"] & \pi_k \operatorname{O}(k+2) \arrow[r, "\approx"] \arrow[d, phantom, "\parallel"] & \pi_k \operatorname{O}(k+3) \arrow[r, "\approx"] \arrow[d, phantom, "\parallel"] & {} \\ \pi_{k+1} \operatorname{BO}(k+1) \arrow[r, two heads] & \pi_{k+1} \operatorname{BO}(k+2) \arrow[r] & \pi_{k+1} \operatorname{BO}(k+3) \arrow[r, "\approx"] & {} \end{tikzcd}$$

For example,

$$\begin{tikzcd} \pi_1 \operatorname{O}(2) \arrow[r] & \pi_1 \operatorname{O}(3) \arrow[r, "\approx"] & \pi_1 \operatorname{O}(4) \arrow[r] & {} \\ \mathbb{Z} \arrow[r] & \mathbb{Z}_2 & = & \mathbb{Z}_2 \end{tikzcd}$$

Corollary: $\pi_2 \operatorname{O}(k) = 0$ for $k \gg 0$.

Corollary 101. Let B be CW complex.

a) $\begin{tikzcd} \xi = \mathbb{R}^n \arrow[r] & E \arrow[d, "n > \dim B."] \\ & B \end{tikzcd}$

$\implies \exists$ nowhere zero secttion ($\iff \xi = \alpha \oplus \epsilon$).

b)
$$\begin{tikzcd} \xi, \eta = \mathbb{R}^n \arrow[r] & E \arrow[d, "n > \dim B."] \\ & B \end{tikzcd}$$

$\xi \oplus \epsilon \cong \eta \oplus \epsilon$ [stability isomorphism] $\implies \xi \cong \eta$ isomorphism.

Now we can define stably orthonormal group:

$$\mathrm{O} = \mathrm{colim}_{n\to\infty} O(n) (= \bigcup_n O(n) \text{ with topology})$$

Then $\pi_k \mathrm{O} = \pi_k \mathrm{O}(n)$ for $n \geq k+2$.

Then we have Bott periodicity

$$\pi_k \mathrm{O} = \begin{cases} \mathbb{Z}_2, & \text{if } k \equiv 0(8); \\ \mathbb{Z}_2, & \text{if } k \equiv 1(8); \\ 0, & \text{if } k \equiv 2(8); \\ \mathbb{Z}, & \text{if } k \equiv 3(8); \\ 0, & \text{if } k \equiv 4(8); \\ 0, & \text{if } k \equiv 5(8); \\ 0, & \text{if } k \equiv 6(8); \\ \mathbb{Z}, & \text{if } k \equiv 7(8). \end{cases}$$

$$\pi_k U = \begin{cases} 0, & \text{if } k \equiv 0(2); \\ \mathbb{Z}, & \text{if } k \equiv 1(2). \end{cases}$$

For $k \leq 7$, the generators are all Hopf bundles over S^{k+1}. There are 4 hopf bundles (reals, complex, quarternions, octonions) and they correspond to the non-zero $\pi_k \mathrm{O}$.

Canonical example: $k = 1$.

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \ (z_1, z_2) \arrow[d] & \\ & \mathbb{C}P^1 \ [z_1 : z_2] & \cong \quad S^2 \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & E(\gamma^1) \arrow[d] \\ & \mathbb{C}P^1 \end{tikzcd}$$

$$\begin{tikzcd} S^3 \arrow[d] & (z_1, z_2) \\ \mathbb{C} \cup \infty & z_1/z_2 \end{tikzcd}$$

Theorem 102 (Splitting Principle)**.** We can have splitting principles for real bundles $\xi = \mathbb{R}^n \to E \to B$ or complex bundles $\mathbb{C}^n \to E' \to B'$.

Assume B, B' are CW. Splitting principle says $\exists$ maps $F \xrightarrow{f} B, F' \xrightarrow{f'} B'$ such that:

1) $f^*E = L_1 \oplus \cdots \oplus L_n$ and $f'^*E' = L'_1 \oplus \cdots \oplus L'_n$, i.e. direct sum of line bundles.
2) These maps are cohomology injections: $f^* : H^*(B; \mathbb{F}_2) \rightarrowtail H^*(F; \mathbb{F}_2), f'^* : H^*(B'; \mathbb{Z}) \to H^*(F'; \mathbb{Z})$.

idea: We can pretend every vector bundle is a sum of line bundle.

For existence of SW (and chern) classes:

Instead of Steenrod squares, we can try to take $f^* \mathrm{w}(E) = \mathrm{w}(L_1) \cdots \mathrm{w}(L_n)$.

These are just line bundles so we can define them by orientations.

Wednesday, 12/3/2025

Theorem 103 (One Step Splitting Principle)**.** $\exists f : P \to B, f' : P' \to B'$ such that:

1) $f^*E = L_1 \oplus E_1, (f')^*E = L'_1 \oplus E'_1$.
2) $H^*(f, \mathbb{F}_2), H^*(f; \mathbb{Z})$ are injective.

One step splitting principle implies splitting principle by induction.

P will be the projective bundle associated to B.

If V is a vector space we have $P(V) = $ lines in $V = \mathrm{Gr}_1(V)$.

Then we have projective bundles:

$$\begin{tikzcd} \mathbb{R}P^{n-1} \arrow[r] & P(E) \arrow[d] & = & \bigcup P(E_b) = E_0/e \sim \lambda e : \lambda \neq 0 \\ & B & & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C}P^{n-1} \arrow[r] & P(E') \arrow[d, "f'"] \\ & B' \end{tikzcd}$$

We have the tautological line bundle:

$$\begin{tikzcd} \mathbb{R} \arrow[r] & L_1 \arrow[d] & = & \{(l, e) \mid e \in l\} \subset P(E) \times E \\ & P(E) & & \end{tikzcd}$$

$$\begin{tikzcd} L_1 \arrow[dr] & \subset & f^*E \arrow[dl] \\ & P(E) & \end{tikzcd}$$

Assume B, B' are CW. Then $f^*E = L_1 \oplus (L_1^\perp)$.

Theorem 104 (Leray-Hirsch, See Hatcher)**.** Let $a = \mathrm{w}_1(\gamma') \in H^1(P(E); \mathbb{F}_2)$.

Let $b = e(\gamma^1) \in H^2(P(E'))$

Then $H^*(P(E); \mathbb{F}_2)$ is a free $H^*(B, \mathbb{F}_2)$-mmodule with basis $1, a, a^2, \cdots, a^{n-1}$.

$H^*(PE')$ is a free H^*B-module with basis $1, b, b^2, \cdots, b^{n-1}$

This implies 1-step S.P. f^*, f'^* are injective since $\{1\}$ is linearly independent.

Grothendieck's Definition of SW and Chern Classes

LH $\implies$ a^n = sum of basis elements, b^n = sum of basis elements.

$a^n = \sum_{i=1}^n f^*(a_i) a^{n-i}$

$b^n = \sum_{i=1}^n f'^*(b_i) b^{n-i}$

Define $\mathrm{w}_i\, E = a_i \in H^1(B; \mathbb{F}_2)$.

$c_i E' = -b_i \in H^{2i}(B'; \mathbb{Z})$.

Back to the splitting principle. What are F and F'?

Flags

Suppose we have vector spae V where $\dim V = n$.

Definition (Flag)**.** $F(V) = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V\}$

If V has an inner product then $F(V) \cong F_0(V) = \{V = L_1 \oplus \cdots \oplus L_n\}$ where L_i are orthogonal lines.

$F = F(E), F' = F'(E)$.

Why are SW classes $\mathbb{F}_2$ coefficient but Chern class $\mathbb{Z}$-coefficient

This boils down to $\mathrm{O}(n)$ vs $\mathrm{U}(n)$.

We have:

$$(\mathbb{Z}_2)^n \hookrightarrow \mathrm{O}(n)$$

$$(S^1)^n \hookrightarrow \mathrm{U}(n)$$

Then,

$$\begin{tikzcd}
E(\gamma^1)\times\cdots\times E(\gamma^1) \arrow[rrr] \arrow[d] & & & E(\gamma^n) \arrow[d] \\
\mathbb{R}P^\infty\times\cdots\times\mathbb{R}P^\infty & = & (B\mathbb{Z}_2)^n = B(\mathbb{Z}_2)^n \arrow[r, "g"] & \mathrm{BO}(n) = \mathrm{Gr}_n\,\mathbb{R}^\infty
\end{tikzcd}$$

$$\begin{tikzcd}
E(\gamma^1)\times\cdots\times E(\gamma^1) \arrow[rrr] \arrow[d] & & & E(\gamma^n) \arrow[d] \\
\mathbb{R}P^\infty\times\cdots\times\mathbb{R}P^\infty & = & (B\mathbb{S}^1)^n = B(S^1)^n \arrow[r, "g"] & \mathrm{BU}(n) = \mathrm{Gr}_n\,\mathbb{C}^n
\end{tikzcd}$$

Theorem 105 (Borel)**.**

$$H^*(\mathrm{BO}(n);\mathbb{F}_2) \overset{g^*}{\rightarrowtail} H^*(\mathbb{R}P^\infty\times\cdots\times\mathbb{R}P^\infty;\mathbb{F}_2)$$

$\operatorname{im} g^* = \mathbb{F}_2[a_1,\cdots,a_n]^{S_n}$.

$$H^*(\mathrm{BU}(n);\mathbb{Z}) \xrightarrow{\approx} \mathbb{Z}[a_1,\cdots,a_n]^{S_n}$$

This gives us another definition of SW classes and chern class.

$$g^*\,\mathrm{w}_i(\gamma^n) = \sigma_i(a_1,\cdots,a_n) = \sigma_i(\mathrm{w}_1(\gamma^1),\cdots,\mathrm{w}_n(\gamma^1))$$

$$(g')^*c_i(\gamma^n) = \sigma_1(b_1,\cdots,b_n)$$

Monday, 12/8/2025

Chern Classes MS Ch13-14

Recall $\mathbb{C}$-vector bundles:

$$\omega = \begin{tikzcd}
\mathbb{C}^n \arrow[r] & E \arrow[d] \\
& B
\end{tikzcd}.$$

This corresonds to a $\mathbb{R}^{2n}$-bundle with acomplex structure:

$$\begin{tikzcd}
\mathbb{R}^{2n} \arrow[dr] & & & \mathbb{R}^{2n} \arrow[dl] \\
& E \arrow[r, "J"] \arrow[dr] & E \arrow[d] & \\
& & B &
\end{tikzcd}$$

Where $J^2 = -\mathrm{Id}$

Open $U \subset \mathbb{C}^n$ then $TU \cong U \times \mathbb{C}^n$.

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(t \mapsto x + tv)\big|_{t=0} \leftarrow\!\shortmid (x, v)$$

$$\begin{tikzcd}
U \times \mathbb{C}^n \arrow[rr, "\approx"] \arrow[dr] & & \pi^{-1}U \arrow[dl] \\
& U &
\end{tikzcd}$$

$J_0 TU \to TU, J_0(x, v) = (x, iv)$

Let $f : U \to U$ where $U \subset \mathbb{C}^n$.

f is *holomorphic* if $\mathrm{d}f \circ J_0 = J_0 \circ \mathrm{d}f$ [= analytic = Cauchy-Riemann eqn hold]

M a $\mathbb{C}$-manifold of $\dim n$ definitions:

Definition (1)**.** Space M with *holomorphic atlas* $A = \{\phi : V_\phi \to U \subset \mathbb{C}^n\}$ so that $\phi_2 \circ \phi_1^{-1}$ is holomorphic.

Definition (2)**.** M is a manifold of $\dim 2n$ with complex structure $J : TM \to TM$ such that $\forall x \in M \exists$ neighborhood V and a diffeomorphism $\phi : V \to U$ where $\mathrm{d}\phi \circ J = J_0 \circ \mathrm{d}\phi$

Definition (Allmost Complex Manifold)**.** An *almost complex manifold* is a smooth manifold on a smooth structure on its tangent bundle.

Examples: $\mathbb{C}^n$ is a complex manifold.

$\mathbb{C}P^n$ are complex manifolds.

Higher dimension torii: $\mathbb{C}^n / \langle \mathbb{Z}^n, i\mathbb{Z}^n \rangle$ are complex manifolds.

$\mathbb{C}P^1 = S^2$ are complex manifolds.

Odd dimensional spheres cannot have complex structures.

Question: When do even dimensional spheres have complex/almost complex structures?

S^4, S^{2n} for $2n > 6$ don't have almost complex structures.

S^6 has almost complex structure.

Axioms:

1) $C_i(\omega) = H^{2i}(B; \mathbb{Z}), C_0(\omega) = 1, C_1(\omega) = 0$ for $i > n$.
2) $C_i\left(f^* \omega\right) = f^* c_i \omega$.
3) $C_k(\omega \oplus \eta) = \sum_{i+j} c_i(\omega) \cup c_j(\eta)$
4) $c_1(\gamma') = -u_{\mathbb{C}P^1} \in H^2(\mathbb{C}P^1)$.

These are called Hopf bundlles

Also $4'$: $c_n(\omega) = e(\omega_{\mathbb{R}})$

$\mathbb{C}$-v.s. maps to oriented v.s.: $V \to V_{\mathbb{R}}$.

$(e_1, \cdots, e_n) \mapsto (e_1, ie_1, e_2, ie_2, \cdots, e_n, ie_n)$.

$$\begin{tikzcd}
 & \mathbb{R} & \\
M_n\mathbb{C} \arrow[ur, "\det\cdot\det"] \arrow[rr] \arrow[d, equal] & & M_{2n}\mathbb{R} \arrow[ul, "\det"'] \arrow[d, equal] \\
\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \arrow[rr] & & \mathrm{End}_{\mathbb{R}}
\end{tikzcd}$$

So $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{n,+}(\mathbb{R})$

Theorem 106. $H^*(G_n\mathbb{C}^\infty) = \mathbb{Z}[c_1(\gamma^n), \cdots, c_n(\gamma^n)]$. Algebraically independent.

Existence of Chern Classes

1) Grothendieck: $a \in H^2P(E), a^{n+1} = -\sum$ chern classes a^i.

2) Borel: $$\begin{tikzcd}
\gamma^1 \arrow[d] & & & & \gamma^1 \arrow[d] & \\
\mathbb{C}P^\infty & \times & \cdots & \times & \mathbb{C}P^\infty \arrow[r, "c"] & Gr_n\mathbb{C}^\infty
\end{tikzcd}.$$

   $c^* : H^*Gr_n(\mathbb{C}^\infty) \to$ $mathbbZ[a_1, \cdots, a_n]^{S_n}$. Then $c_i \leftrightarrow \sigma_1(a_1, \cdots, a_n)$.

3) MS: $c_i(\omega) = \begin{cases} (\pi_0^*)^{-1} c_i(\omega_0), & \text{if } i < n; \\ e(\omega_{\mathbb{R}}), & \text{if } i = n. \end{cases}$

   Assume inductively that $c_i\phi$ is defined for rank $\phi < n$.

   $$\omega_0 = \begin{tikzcd}
\mathbb{C}^{n-1} \arrow[r] & E_0 \arrow[d, "\pi_0"] & = & E - z(B) \\
 & B & &
\end{tikzcd}$$

   $\pi_0^*\omega$ has nowhere zero section $s : E_0 \to E_0 \times_B E_0 = \pi_0^*\omega, v \mapsto (v, v)$.
   $\epsilon^1 \subset \pi_0^*\omega$.
   Then $\omega_0 = \pi_0^*\omega/\epsilon^1$.

Remark: if ω has a metric then $\pi_0^*\omega = \epsilon^1 \oplus (\epsilon^1)^\perp$

Also, $E_0 \xrightarrow{f} P(E)$ then $f^*\gamma^1$ is trivial.

Wednesday, 12/10/2025

Chern-Weil Theory

$$c_1 L = \left[\frac{i}{2\pi}\Omega\right] \in H^2_{DR}M$$

Ω is curvature of a metric connection

Complex Theory of Connection

Let
$$\begin{tikzcd} \mathbb{C}^n \arrow[r] & E \arrow[d] \\ & M \end{tikzcd}$$
be a smooth $\mathbb{C}$ v.b. over a smooth (real) manifold.

$\Gamma(E) =$ smooth section
$$\begin{tikzcd} E \arrow[d] \\ M \arrow[u, bend right, dotted, "s"'] \end{tikzcd}$$

Let $\Omega^i(M; E)$ be i-forms with values in E.

$\Omega^0(M; E) = \Gamma(E)$.

$\Omega^1(M; E) = \Gamma(T^*M \otimes_{\mathbb{R}} E) = \Gamma(\mathrm{Hom}(TM, E))$.

$\mathbb{C}^\infty M =$ smooth $M \to \mathbb{R}$.

$\Omega^i(M; E) = \Gamma(\Lambda^i T^*M \otimes E)$

Definition. A connection on E is a $\mathbb{C}$-linear map ∇ akin to derivative given by:

$$\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$$

which satisies the Liebniz law:

$$\nabla(fs) = df \otimes s + f\nabla s$$

Where $s \in \Gamma E, f \in \mathbb{C}^\infty M$.

For $X \in \Gamma(TM)$, section of tangent bundle is a vector field,

$$\nabla_X \Gamma(E) \to \Gamma(E)$$

is kind of a 'directional derivative':

$$\nabla_x s \coloneqq \nabla(s)X$$

Definition. A *hermitian metric* on E is a function $\langle , \rangle : E \times_M E \to \mathbb{C}$. It is a fancy notation for the pullback: given two points in a fiber we want a complex number. It is a $\mathbb{C}$-inner product on fibers. The inner product has to be hermitian.

Definition (Metric Connection)**.** By picking two sections s, t note that $\langle s, t\rangle$ is a function $M \to \mathbb{C}$.

$$d\langle s, t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle \in \Omega^1(M; \mathbb{C})$$

Local View

Lemma 107 (1)**.** Consider trivial bundle $(U \times \mathbb{C}^n)$

A connection is determined by matrix $\omega_{ij} \in M_n(\Omega^1(M;\mathbb{C})) = \Omega^1(M; M_n\mathbb{C})$.

In case of a metric connection, (ω_{ij}) is skew hermitian.

For $n = 1$ in the metric case $\omega \in \Omega^1(M; i\mathbb{R})$. In this case, locally, this is given by just a one-form.

$\omega =$ connection 1-form.

Proof. Let $s_1, \cdots, s_n$ be linearly independent section (orthonormal in metric case):

$$\nabla(s_i) = \sum \omega_{ij} \otimes s_j$$

$$\nabla(f_1 s_1 + \cdots + f_n s_n) = \sum df_i \otimes s_i + f_i \nabla s_i$$

In the metric case since s_i are orthonormal, $0 = d\langle s_i, s_j\rangle = \langle \nabla s_i, s_j\rangle + \langle s_i, \nabla s_j\rangle = \omega_{ij} + \overline{\omega}_{ji}$. ☐

Lemma 108 (2)**.** Every bundle has a connection.

Proof. Take a partition of unity $(\{U_\alpha\}, \lambda_\alpha)$ on M so that $E\big|_{U_\alpha}$ are trivial. Take $\nabla = \sum \lambda_\alpha \nabla_\alpha$. ☐

Curvature of Connection

Consider $\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & E \\ & & \big\downarrow \\ & & M \end{array}$ with metric.

Curvature of connection:

$$\Omega(\nabla) = \Omega \in \Omega^2(M; \mathrm{Hom}(E, E))$$

If ∇ is metric then $\Omega \in \Omega^2(M; U_n)$

Local Def: $\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj}$.

Global Def 1: $\Omega_{x,y}(s) = \nabla_x \nabla_y s - \nabla_y \nabla_x s - \nabla_{[x,y]} s$

Global Def 2: $\Omega = \nabla \circ \nabla$.

Now we look at line bundles. Suppose we have a smooth line bundle $\begin{array}{ccc} \mathbb{C} & \longrightarrow & L \\ & & \big\downarrow \\ & & M \end{array}$ with a metric.

Locally a connection is given by 1-form $\omega \in \Omega^1(U; i\mathbb{R})$.

$\Omega = d\omega - \omega \wedge \omega \in \Omega^2(M; i\mathbb{R})$.

Facts:

1) 1. $d\Omega = 0$ curv. closed

$d\Omega = d(d\omega) - (d\omega \wedge \omega) + \omega \wedge d\omega = 0.$

$[\frac{1}{i}\Omega] \in H^2_{DR}M = H^2(M;\mathbb{R})$

2) $\Omega(\nabla) - \Omega(\nabla') = d\beta$

So, $[\frac{1}{i}\Omega]$ is independent of connection.

3) $[\frac{1}{i}\Omega]$ is a characteristic class.

$\implies [\frac{1}{i}\Omega] = a(c_1(L)) \in H^2(M;\mathbb{R})$ for some $a \in \mathbb{R}$.

4) $a = \frac{1}{2\pi}$. Compute for Hopf bundle:

$$\begin{tikzcd} \mathbb{C} \arrow[r] & E(\gamma^1) \arrow[d] \\ & \mathbb{C}P^1 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^2 \arrow[r] & E \arrow[d] \\ & S^2 \end{tikzcd}$$

Use Gauss Bonnet.