# M522 Topology II

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## Monday, 1/13/2025

## Singular Homology and CW Complexes

We want to talk about the Homology of a space $X$.

**Definition** (Homology)**.** Let $X$ be a topological space. Consider the sequence of abelian groups:

$$H_0X, H_1X, H_2X, \cdots$$

These are the homemorphism invariants.

For example, consider the 2-torus $T^2$ and the 2-sphere $S^2$. They are not homeomorphic, we can see that from their fundamental groups.
$H_1T^2 = \mathbb{Z} \oplus \mathbb{Z}$.
$H_1S^2 = 0$.
$\therefore S^2 \not\cong T^2$
Do note that, even if all elements from the sequence are isomorphic the spaces might not be isomorphic!
Some application: see Davis and Kirk "Homology Greatest Hits".
Knot theory seems very intuitive but proving statements is very troublesome. For example, how do you prove that the trefoil and the unknot are not the same?

**Theorem 1** (Brouwer's Fixed Point Theorem)**.** Every $f : D^n \to D^n$ has a fixed point.

**Theorem 2** (Euler's Formula)**.** For every 'triangulation' of $S^2$ we have:

$$v - e + f = 2 = \chi(S^2)$$

$\chi$ denotes the Euler Characteristic.
eg pyramid $4 - 6 + 4 = 2$, triangulated bipyramid $5 - 9 + 6 = 2$, cube $8 - 12 + 6 = 2$.

**Theorem 3** (Hairy Ball Theorem)**.** $\nexists f \colon S^2 \to S^2$ s.t. $\forall x \in S^2, x \cdot f(x) = 0$.
So you can't comb the hairy ball.

**Theorem 4** (Jordan Curve Theorem)**.** The complement of a closed curve in plane has two components.

**Theorem 5** (Brouwer's Theorem on Invariance of Domain)**.** $m \neq n \implies \mathbb{R}^m \not\cong \mathbb{R}^n$.
Consider open $U \subset \mathbb{R}^n$ [a domain] and let $f$ be a continuous injection $f \colon U \to \mathbb{R}^n$.
Then $f(U)$ is open in $\mathbb{R}^n$.

### Variants of Homology

| | defined for | |
|---|---|---|
| Singular Homology | Top Spaces | Easy to define but hard to compute |
| Simplicial homology | simplicial complexes and $\Delta$-complexes | Easy to define and compute but difficult to show homeo inv. |
| Cellular homology | CW-complexes | hard to define, easy to compute, flexible. |

Table 1: Variants of Homology

# Definition of Singular Homology

**Definition** (Standard $n$-simplex)**.**

$$\Delta^n = \left\{ (t_0, \cdots, t_n) \mid \sum_i t_i = 1, 1 \geq t_i \geq 0 \right\} \subset \mathbb{R}^{n+1}$$

$$= \left\{ \sum_i t_i \underline{e}_i \mid 1 \geq t_i \geq 0, \sum_i t_i = 1 \right\}$$

$$= \text{convex hull of } \{\underline{e_0}, \cdots, \underline{e_n}\}$$

Recall that convex hull is the intersection of all convex sets containing the original set.

# Singular $n$-simplex in $X$

$n$-simplices are images of standard simplices under continuous maps.
They are defined by a continuous map $\sigma : \Delta^n \to X$.
We define singular $n$-chains $S_n X$. These are free abelian groups with $\mathbb{Z}$-basis the singular $n$-simplicies in $X$.
A typical element will be a finite sum:

$$n_1\sigma_1 + \cdots + n_k\sigma_k \in S_n X$$

Where $\sigma_i : \Delta^n \to X$.
Note: Davis and Kirk uses $S_n X$, Hatcher uses $C_n X$.
For example, let $X$ be the punctured plane $X = \mathbb{R}^2 - \{0\}$.
$\sigma_1 + \sigma_2 - \sigma_3 \in S_1 X$.
This is an example of a special 1-chain callled the 1-cycle.
<u>Goal</u>: Define a boundary map $\partial_n : S_n X \to S_{n-1} X$ [Read Davis Kirk].
Then, $H_n$ is given by the quotient map:

$$H_n = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{n\text{-cycles}}{n\text{-boundaries}}$$

# Wednesday, 1/15/2025

<u>Goal</u>: We want to define a homomorphism called a <u>boundary map</u>.

$$\partial_n : S_n X \to S_{n-1} X$$

We start with the $j$'th face map.

$$\delta_j = \delta_j^n : \Delta^{n-1} \to \Delta^n$$

We have the map of barycentric coordinates:

$$(t_0, \cdots, t_{n-1}) \mapsto (t_0, \cdots, t_{j-1}, 0, t_j, \cdots, t_{n-1})$$

The $j$'th face map of $\sigma$ is given by precomposing $\delta_j$:

$$\sigma \circ \delta_j : \Delta^{n-1} \to X$$

**Definition.** The boundary $\partial_n \sigma = \sum_{j=0}^{n} (-1)^j \sigma \circ \partial_j$.
We can extend this definition tp $S_n$ by linearity.

$$\partial_n \left( \sum_j n_j \sigma_j \right) = \sum_j n_j \partial_n \sigma_j$$

Let $\sigma : \Delta^2 \to X$.
Then, $\partial_2 \sigma = \sigma \circ \delta_0 - \sigma \circ \delta_1 + \sigma \circ \delta_2$.

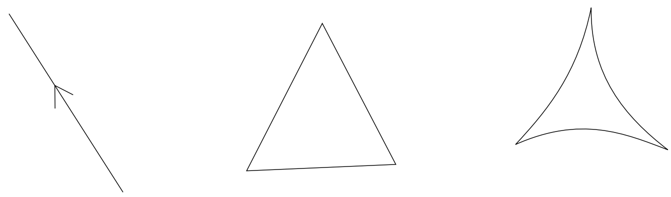


Figure 1: Boundary Map

$\sigma : \Delta^1 \to X$
$\partial \sigma = \sigma(e_1) - \sigma(e_0) = c_{\sigma(e_1)} = c_{\sigma(e_0)}$, endpoint - starting point.

**Lemma 6.** $\partial_{n+1} \circ \partial_n = 0$.

$$\begin{tikzcd} S_{n+1}X \arrow[r, "\partial_{n+1}"] \arrow[rr, bend right, "0"'] & S_nX \arrow[r, "\partial_n"] & S_{n-1}X \end{tikzcd}$$

This is the reason for $-$ signs.

Then, we have,

$$\begin{array}{ccccc} \operatorname{im} \partial_{n+1} & \subset & \ker \partial_n & \subset & S_nX \\ n\text{-boundaries} & & n\text{-cycles} & & n\text{-chains} \end{array}$$

**Definition** (Singular Homology)**.**

$$H_nX = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{\text{cycles}}{\text{boundaries}}$$

*Proof.* We prove the lemma: $\partial_{n-1} \circ \partial_n = 0$.

$$\partial_{n-1}(\partial_n \sigma)$$

$$= \partial_{n-1} \left( \sum_j (-1)^j \sigma(t_0, \cdots, 0, \cdots, t_{n-1}) \right)$$

$$= \sum_{k<j} (-1)^k (-1)^j \sigma(t_0, \cdots, 0, \cdots, 0, \cdots, t_n) \, 0 \text{ s in } k\text{'th and } j\text{'th slots}$$

$$+\sum_{k>j}(-1)^{k-1}(-1)^j\sigma(t_0,\cdots,0,\cdots,0,\cdots,t_n)\,0\text{ s in }k\text{'th and }j\text{'th slots}$$

$$=0$$

$\square$

**Remark.** 1) $H_nX$ is defined for any topological space $X$ and $n \geq 0$.

2) $X \cong Y \implies H_nX \cong H_nY$.

3) Big and Formula Construction.

4) Unclear how to compute.

Answer to the question: What is $H_nX$:

$$H_*X = \{H_0X, H_1X, H_2X, \cdots\}$$

is a graded abelian group. $H_kX$ individually are abelian groups.

**Lemma 7** (Lemma 1)**.**

$$H_n(pt) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma 8** (Lemma 2)**.** If $X$ has path-components $\{X_\alpha\}_{\alpha\in I}$, then,

$$H_nX = \bigoplus_{\alpha\in I} H_n(X_\alpha)$$

**Lemma 9** (Lemma 3)**.** a) $H_0X \cong \bigoplus_I \mathbb{Z} = \mathbb{Z}^{\#\text{ of path component}}$

b) $X$ is path-connected, then $H_0X \cong \mathbb{Z}$.

Recall:

**Definition.** $X$ is path-connected if $\forall a, b \in X, \exists\gamma : [0,1] \to X$ such that $\gamma(0) = a, \gamma(1) = b$

**Definition.** A maximal path-connected subset of $X$ is path-component.

**Corollary 10.** Homology of rational numbers is isomorphic to the homology of integers:

$$H_*\mathbb{Q} \cong H_*\mathbb{Z} = (\mathbb{Z}^\infty, 0, 0, \cdots)$$

But $\mathbb{Q} \not\cong \mathbb{Z}$.

# Friday, 1/17/2025

Recall:
$H_nX = \frac{\ker(\partial_n)}{\text{im}(\partial_{n+1})} = \frac{\text{cycles}}{\text{boundary}} \in$ homology class.
We are looking for two cycles that belong to the same homology class.
So, we want cycles $z_1 \neq z_2$ which are homologous so that $z_1 - z_2$ is a boundary. This implies their homology classes are equal: $[z_1] = [z_2]$.

# Algebra

**Definition** (Chain Complex)**.** A chain complex $C_\bullet$ is a sequence:

$$C_* = \{C_0, C_1, C_2, \cdots\}$$

of abelian groups with $\partial_n : C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$.
It looks like the following:

$$\cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

so that the composition of any two consecutive maps is 0. By conventions, $\partial_0 = 0 : C_0 \to 0$.

Then, $C_\bullet = \{C_*, \partial_*\}$.

**Definition** (Homology)**.**

$$H_n C_\bullet = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{Z_n}{B_n}$$

Here, $C_n = n$-chain.
$Z_n = \ker \partial_n, n$-cycles
$B_n = \operatorname{im} \partial_{n+1}, n$-boundaries.
eg. $S_\bullet X = \{S_* X, \partial_*\}$ is a singular chain complex of $X$.

L1:

$$H_n \text{pt} = \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_*(\text{pt}) = \{\mathbb{Z}, 0, 0, \cdots\}$$

*Proof.* $\forall n, \exists! \sigma_n : \Delta^n \to \text{pt}$.
Then, $\partial_1 \sigma_1 = \sigma_1 \circ \delta_0 - \sigma_1 \circ \delta_1$.
$\delta_0(t_0) = (0, t_0), \delta_1(t_0) = (t_0, 1)$.
Thus, $\partial_1 \sigma_1 = \sigma_1 \circ \delta_0 - \sigma_1 \circ \delta_1 = (1-1)\sigma_0 = 0$
$\partial_2 \sigma_2 = \sigma_2 \circ \delta_0 - \sigma_2 \circ \delta_1 + \sigma_2 \circ \delta_2 = (1-1+1)\sigma_1 = \sigma_1$.
$\partial_n \sigma_n = \begin{cases} 0, & \text{if } n \text{ odd;} \\ 1, & \text{if } n \text{ even.} \end{cases}$
$S_* X$ :

$$\longrightarrow \mathbb{Z}\sigma_2 \longrightarrow \mathbb{Z}\sigma_1 \longrightarrow \mathbb{Z}\sigma_0$$

$$\sigma_2 \longmapsto \sigma_1 \longmapsto 0$$

$$\cong \qquad \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$H_0 \text{pt} = \mathbb{Z}/00 = \mathbb{Z}$.
$H_1 \text{pt} = \mathbb{Z}/\mathbb{Z} = 0$
$H_2 \text{pt} = 0/0 = 0$. ∎

L2: If $\{X_\alpha\}_{\alpha \in I}$ are path components of $X$ then,

$$H_n X = \bigoplus_I H_n X_\alpha$$

*Proof.* $\sigma : \Delta^n \to X \overset{\Delta^n \text{p.c.}}{\Longrightarrow} \sigma(\Delta^n)$ p.c.
$\Longrightarrow \exists! \alpha$ such that $\sigma(\Delta^n) \subset X_\alpha$.
Also $(\sigma \circ \delta_j)(\Delta^{n-1}) \subset X_\alpha$
$S_* X = \bigoplus_I S_* X_\alpha$ ∎

Augmentation
$\varepsilon : S_0X \to \mathbb{Z}$
$\varepsilon(\sum_i n_i\sigma_i) \coloneqq \sum_i n_i$
$\varepsilon \circ \partial_1(\sigma) = \varepsilon(\sigma \circ \delta_0 - \sigma \circ \delta_1) = 1 - 1 = 0$
Thus $\operatorname{im}\partial_1 \subset \ker \mathscr{E}$
Thus, $\exists \overline{\varepsilon} : H_0X \to \mathbb{Z}$
$\overline{\varepsilon}\left[\sum_i n_i\sigma_i\right] = \varepsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i$.
L3:

1) If $X$ is path connected then,

$$\overline{\varepsilon} : H_0X \xrightarrow{\cong} \mathbb{Z}$$

2) If $\{X_\alpha\}_{\alpha \in I}$ are path components of $X$ then,

$$H_0X = \bigoplus_I H_0X_\alpha = \mathbb{Z}^{\#\text{ of p.c. of } X}$$

*Proof.* 1) Need to show $\ker\varepsilon \subset \operatorname{im}\partial_1$.
Choose base point $x_0 \in X$.
Suppose $\varepsilon\left(\sum_i n_i\sigma_i\right) = 0$.
Choose path $\gamma_i : \Delta^1 \to X$ such that $\gamma_i(\underline{e}_1) = \sigma_i(\underline{e}_0), \gamma_0(e_0) = x_0$.
$\partial_1\left(\sum_i n_i\gamma_i\right) = \sum_i n_i\sigma_i - \sum_i n_i \mathrm{cons}_{X_0} = \sum_i n_i\sigma_i$

$\square$

## $\Delta$-complex (p.102-104 of Hatcher)

eg torus

$$\longrightarrow$$

$1 = \#$ of vertices
$3 = \#$ of edges
$2 = \#$ of faces
$\Delta_2T \to \Delta_1T \to \Delta_0T$
$\cong \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}^1$

## Wednesday, 1/22/2025

**Definition** (Simplex)**.** Let $v_0, \cdots, v_n \in \mathbb{R}^n$.

$$[v_0, \cdots, v_n] = \left\{\sum_i t_iv_i \mid \sum_i t_i = 1, 1 \leq t_i \leq 0\right\}$$

If $v_0 - v_1, \cdots, v_0 - v_n$ are linearily independent, then $[v_0, \cdots, v_n]$ is a geometric $n$-simplex.

If vertices are ordered,

$$\sigma_{[v_0,\cdots,v_n]} : \Delta^n \to [v_0, \cdots, v_n]$$

$$\sum_i t_ie_i \mapsto \sum_i t_iv_i$$

Note that,

$$\delta_n\sigma_{[v_0,\cdots,v_n]} \equiv \sum_j (-1)^j \sigma_{[v_0,\cdots,\widehat{v_j},\cdots,v_n]}$$

$\Delta^n$ breaks up into boundary and interior.

$$\Delta^n = \partial\Delta^n \cup \overset{\circ}{\Delta^n}$$

$$\partial\Delta^n = \left\{ \sum_i t_i e_i \mid \text{some } t_i = 0 \right\}$$
$$\overset{\circ}{\Delta^n} = \left\{ \sum_i t_i e_i \mid t_i \neq 0 \text{ for all } i \right\}$$

**Definition.** A $\Delta$-complex is a space $X$ with:

$$\{\sigma_\alpha : \Delta^n \to X\} \text{ simplices}$$

such that:

i) $\sigma_\alpha\big|_{\overset{\circ}{\Delta^n}}$ is injective.

   $\forall x \in X, \exists!$ s.t. $x \in \sigma_\alpha(\overset{\circ}{\Delta^n})$. Images of interiors partition $X$.

ii) $\forall \alpha, \forall j, \exists \beta$ such that:

$$\sigma_\alpha \circ \delta_j = \sigma_\beta$$

   Faces of simplices are simplices.

iii) $A \subset X$ open $\iff \forall \sigma_\alpha, \sigma_\alpha^{-1} A$ is open in $\Delta^n$. "Weak Topolofy".

Hatcher says, one way to look at this is by taking a quotient of a disjoint union. We can consider:

$$\frac{\Delta^0 \sqcup \Delta^1 \sqcup \Delta^1 \sqcup \Delta^1 \sqcup \Delta^2 \sqcup \Delta^2}{\sim}$$

**Definition** (Simplicial Chain Complex)**.** $\Delta_n X =$ free abelian group on $n$-simplices:

$$\Delta_{n+1} X \xrightarrow{\partial_{n+1}} \Delta_n X \to \Delta_{n-1} X \to$$

Subcompplex $\Delta_* X \subset S_* X$

$$\begin{array}{ccc} \Delta_n X & \xrightarrow{\partial} & \Delta_{n-1} X \\ \big\downarrow & & \big\downarrow \\ S_n X & \xrightarrow{\partial} & S_{n-1} X \end{array}$$

The diagram commutes.
Simplicial Homology:

$$H_*^\Delta X \coloneqq H_*(\Delta_* X)$$

$$H_*^\Delta X \xrightarrow{\cong} H_* X$$

Consider the 2-torus with "ordered" vertices:

$$\partial U = b - c + a$$

$$\partial L = a - c + b$$

$$\partial a = 0v, \partial b = 0v, \partial c = 0v$$

$$\Delta_2 T \to \Delta_1 T \to \Delta_0 T \to 0$$

Then, $\Delta_2 T$ has basis $\mathbb{Z}U \oplus \mathbb{Z}L$
$\Delta_1 T$ has basis $\mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$
$\Delta_0 T$ has basis $\mathbb{Z}v$
Now, $H_0^\Delta T = \frac{\mathbb{Z}v}{0} \cong \mathbb{Z}$
$H_2^\Delta T = \frac{\mathbb{Z}(U-L)}{0} \cong \mathbb{Z}$.
We only have $U - L$ since $\partial(n_1 U + n_2 L) = 0 \implies n_1(b-c+a) + n_2(a-c+b) = 0 \implies n_1 = -n_2$
$H_1^\Delta T = \frac{\mathbb{Z}(a,b,c)}{\mathbb{Z}(a-c+b)} = \frac{\mathbb{Z}(a,b,a-c+b)}{\mathbb{Z}(a-c+b)} \cong \mathbb{Z}^2$.
We can have a basis for homology since they are free abelian.
Basis for $H_0^\Delta T$ is $[v]$
Basis for $H_1^\Delta T$ is $[a], [b]$.
Basis of $H_2^\Delta T$ is $[U-L]$.
Matrix POV on $H_*^\Delta T$

$$\Delta. \cong \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}} \mathbb{Z}$$

Integral row and column operations:

- Switch two rows (or columns)
- Add multiples of a row (or column) to another row (or column)
- Multiply row (or column) by $\pm 1$

These correspond to basis changes in the domain and codomain.
If matries $A$ and $B$ are equivalent ($A \sim B$) it implies:

$$\ker A \cong \ker B$$

$$\operatorname{coker} A \cong \operatorname{coker} B$$

Every integral matrix is equivalent to:

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

With $d_1 \mid d_2 \mid d_3 \mid \cdots$
It is a smith normal form.
We have:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is $\partial_2$.
$H_2^\Delta T \cong \mathbb{Z}$
$\operatorname{im} \partial_2 \cong \mathbb{Z}$ and summand of $\Delta_1 T$
$H_1^\Delta T \cong \frac{\mathbb{Z}^3}{\mathbb{Z} \times 0 \times 0} \cong \mathbb{Z}^2$
Exercise: compute kernel and cokernel of: $\begin{bmatrix} 9 & 3 \\ 4 & 2 \end{bmatrix}$
$\begin{bmatrix} 9 & 3 \\ 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 5 & 1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 6 & 0 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$
Thus, $\ker = 0, \operatorname{coker} = \mathbb{Z}/6\mathbb{Z}$.

## Friday, 1/24/2025

Goal: if there is continuous $f : X \to Y$ then we have homomorphism $f_* : H_*X \to H_*Y$.
We think of them in terms of Category Theory.
Setup:
Category $\mathcal{C}$.
Collection of objects $\mathrm{Ob}\,\mathcal{C}$.
$\forall X, Y \in \mathrm{Ob}\,\mathcal{C}$ we have a collection of morphisms $\mathcal{C}(X, Y)$.
$\forall X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$ we have composition law:

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$$

$$(g, f) \mapsto f \circ g$$

Also, $\forall X \in \mathrm{Ob}\,\mathcal{C}, \exists\, \mathrm{id}_X \in \mathcal{C}(X, X)$.
We also have associative law:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

$\forall f \in \mathcal{C}(X, Y), f = \mathrm{id}_Y \circ f = f \circ \mathrm{id}_X$.
For $f \in \mathcal{C}(X, Y)$ we can also write it as $f : X \to Y$ or $X \xrightarrow{f} Y$.
We sometimes call them 'arrows' instead of 'morphisms' to avoid thinking of them as functions.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & Y \arrow[r, "f"] & Z \end{tikzcd}$$

**Definition.** $f : X \to Y$ is an isomorphism if $\exists g : Y \to X$ such that:

$$f \circ g = \mathrm{id}_Y, g \circ f = \mathrm{id}_X$$

We write it as $X \cong Y$ and say $X$ and $Y$ are isomorphic.

Example of Categories:
Set is (sets, functions).
Top is (topological spaces, continuous functions)
Ab is (abelian groups, homomorphisms)
Morphisms need not be functions!!
A group can be viewed as a category with one object. Elements of the group is the set of morphisms, and all morphisms are invertible.
Suppose $G = \{1, T\}$ of order 2. Then, we have:

$$\begin{tikzcd} T \arrow[loop left, "T"'] \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$$

Where $T \circ T = \mathrm{id}$
We let Ch be the category of chain complexes.
The objects will be chain complexes. What are the morphisms?
Recall that Chain complexes are $C_\bullet = (C_*, \partial_*)$ where:

$$\longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow$$

Where $\partial_{n+1} \circ \partial_n = 0$.
Morphisms are given by chain maps.

**Definition** (Chain map)**.** $f_\bullet : C_\bullet \to C'_\bullet$ is a sequence of homomorphisms $f_n : C_n \to C'_n$ such that:

$$f_{n-1} \circ \partial_n = \partial'_n \circ f_n$$

For all $n$.

We have the following commutative diagram:

$$
\begin{tikzcd}
{} \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & C'_{n+1} \arrow[r] & C'_n \arrow[r] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}
$$

For example, if we have $f : X \to Y$ we have the chain map:

$$f_\# = S_\bullet f : S_\bullet X \to S_\bullet Y$$

Given by:

$$(S_\bullet f)\left(\sum_i n_i\sigma_i\right) \coloneqq \sum_i n_i(f \circ \sigma_i)$$

This gives us:

$$
\begin{tikzcd}
S_n X \arrow[r, "\partial_n"] \arrow[d, "S_n f"] & S_{n-1}X \arrow[d] \\
S_n Y \arrow[r, "\partial_n"] & S_{n-1}Y
\end{tikzcd}
$$

**Lemma 11.** A chain map $f_\bullet : C_\bullet \to C'_\bullet$ induces $f_* = H_n(f_\bullet) : H_n C_\bullet \to H_n C'_\bullet$ given by $[x] \mapsto [f_n x]$

**Remark.** elements in $\frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$ can be $[x]$ [equivalence classes] or $x + \operatorname{im} \partial_{n+1}$ [cosets].
We use equivalence classes:
$x \sim x' \iff x - x' \in \operatorname{im} \partial$

*Proof.* $f_n$(cycles) $\subset$ cycles.
$f_n$(boundaries) $\subset$ boundaries.
Recall that cycle is $\ker \partial_n$.
Consider a cycle $X$. Then, $\partial X = 0 \implies f(\partial x) = 0 \implies \partial' f(x) = 0 \implies f(x) \in \ker \partial'$.
Boundary is $\operatorname{im} \partial_{n+1}$.
$f(\partial y) = \partial' f(y) \in \operatorname{im} \partial'_{n+1}$.
Thus we have:

$$\ker \partial_n \to \ker \partial'_n \to \frac{\ker \partial'_n}{\operatorname{im} \partial'_{n+1}}$$

This induces,

$$\frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} \to \frac{\ker \partial'_n}{\operatorname{im} \partial'_{n+1}}$$

□

Now we move on to functors. Functors are an analogy of functions on Categories.
Consider two categories $\mathcal{C}$ and $\mathcal{D}$. We want to define a functor between them.

**Definition.** A functor $F : \mathcal{C} \to \mathcal{D}$ will be an 'assignment' of objects and morphisms.
We have $F : \operatorname{Ob} \mathcal{C} \to \operatorname{Ob} \mathcal{D}$.
$\forall X, Y \in \operatorname{Ob} \mathcal{C}$ we have:

$$F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$$

Then, $F(f \circ g) = F(f) \circ F(g)$.
$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$

So we can $F$ a whole category:

$$X \xrightarrow{f} Y$$

$$F(X) \xrightarrow{F(f)} F(Y)$$

We have the singular functor taking topological spaces to chain complexes. We also have functor taking chain complexes to abelian groups.

$$\mathrm{Top} \xrightarrow{S_\bullet} \mathrm{Ch} \xrightarrow{H_n} \mathrm{Ab}$$

We also have forgetful functor which forgets:

$$\mathrm{Ab} \to \mathrm{Set}$$

$$\mathrm{Ab} \to \mathrm{Group}$$

We have the category Gr of graded abelian groups.
Gr has objects $A_* = \{A_0, A_1, A_2, \cdots\}$ set of abelian groups, and morphisms $A_* \xrightarrow{f_0} B_*$.
Then we can write:

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "S_\bullet"] \arrow[rrr, bend right, "H_n"'] & \mathrm{Ch} \arrow[r, "H_*"] & \mathrm{Gr} \arrow[r, "H_n"] & \mathrm{Ab} \end{tikzcd}$$

**Lemma 12.** Consider a functor $F : \mathcal{C} \to \mathcal{D}$. Then, $X \cong Y \implies F(X) \cong F(Y)$.

Corollary: $X \cong Y$ (homeomorphic) implies $H_n X \cong H_n Y$

*Proof.* $X \cong Y$ implies we have $f, g$ so that $f(X) = Y, g(Y) = X$ so that $f \circ g = \mathrm{id}_Y, g \circ f = \mathrm{id}_X$.
Then, $F(f) \circ F(g) = F(f \circ g) = F(\mathrm{id}_Y) = \mathrm{id}_{F(Y)}$. Similar for $g \circ f$. So, $F(f)$ and $F(g)$ are isomorphisms and thus $F(X)$ and $F(Y)$ are isomorphic. $\square$

# Monday, 1/27/2025

# Homotopy Invariance of Homology

**Definition** (Homotopy)**.** $H : X \times I \to Y \; I = [0,1]$.
Homotpy is a 'path' of map $H_t : X \to Y$ where $t \in [0,1]$ is 'time'. $H_t(x) = H(x,t)$.

**Definition.** $f, g : X \to Y$ are homotopic if there exists $H : X \times I \to Y$ such that $H_0 = f, H_1 = g$.

If they're homotopic we write $f \simeq g$.

**Theorem 13** (Homotopy Theorem)**.**

$$f \simeq g \implies H_* f = H_* g : H_* X \to H_* Y$$

$$f_* = g_*$$

Fact/Exercise: $\simeq$ is a equivalence relation on $\mathrm{TOP}(X,Y)$
$[X,Y] = \mathrm{TOP}(X,Y)/\simeq =$ homotopy classes of map $X \to Y$.
Suppose for $X, Y$, $X \overset{f}{\underset{g}{\leftrightarrows}} Y$ we have $f \circ g \simeq \mathrm{id}_X, X \cong H_* Y$.

For example, $\mathbb{R}^2 - 0 \simeq S^1$. by $x \overset{f}{\mapsto} \frac{x}{|x|}$ and $g =$ inclusion.
Straight line homotopy $tx + (1-t)\frac{x}{|x|}$

**Definition.** $X$ contractible if $X \simeq pt$.
eg $\mathbb{R}^2 \simeq *$

**Definition.** Homotopy Category hTOP.
Objects: Topological Spaces [NOT HOMOTOPY EQUIVALENCE CLASSES]
Morphisms: $\text{hTOP}(X, Y) = [X, Y]$. [These are Equivalence Classes]
Exercise: Composition is well defined. So, $f \simeq f', g \simeq g' \implies f \circ g \simeq f \circ g'$. Then, $[f] \circ [g] := [f \circ g]$

## Isomorphisms in hTOP

$X, Y$ Isomorphic in hTOP $\iff X \simeq Y$ [homotopy equivalence].
So, homotopy theorem says homology factors through homotopy:

$$\begin{tikzcd}
\text{hTop} \arrow[r, "H_*"] & \text{Gr} \\
\text{TOP} \arrow[u] \arrow[ur, "H_*"'] &
\end{tikzcd}$$

Thus, $X \simeq Y \iff H_*X \implies H_*Y$ is really just a consequence of homotopy theorem.

**Definition.** $A \subset X$, then $A$ is a deformation retract of $X$ 'if we can deform all of $X$ into $A$'. Formally,
if $\exists H : X \times I \to X$ such that:
$\forall x \in X, H(x, 0) = x, H(x, 1) \in A$ and $H(a, 1) = a \forall a \in A$.
Suppose $A \simeq X$, $A \overset{i}{\hookrightarrow} X, A \xleftarrow{H_1} X$.
$\text{id}_A = H_1 \circ i, \text{id}_X \underset{H}{\simeq} i \circ H_1$
eg Möbius strip $(X)$ is homotopy equivalent to the cure circle $A$.

**Definition.** $X \subset \mathbb{R}^n$ is start shaped at $p_0 \in X$ if $\forall p \in X, \overline{pp_0} \subset X$.

convex $\implies$ star-shaped $\implies$ contractable (by the straight line homotopy)
We can take $(1-t)p_0 +^\top = H(p, t)$.
eg the $n$-simplex $\Delta^n$ and the prism $\Delta^n \times I$ are star-shaped.

**Theorem 14.** $X$ start shaped $\implies H_*(X) = \mathbb{Z}$ or $0$.

**Corollary 15.** $H_*(\Delta^n) \cong H_*(pt)$
$H_*(\Delta^n \times I) \cong H_*(pt)$.

*Proof.* Wlog $\sigma : \Delta^n \to X$.
Syspension $s\sigma : \Delta^{n=1} \to X$:

$$s\sigma(t_0, \cdots, t_{n+1}) = \begin{cases} (1-t_0)\sigma\left(\dfrac{t_1}{1-t_0}, \cdots, \dfrac{t_{n+1}}{1-t_0}\right), & \text{if } t_0 \neq 1; \\ 0, & \text{if } t_0 = 1. \end{cases}$$

We will show that if $z$ is a $n$-cycle with $n > 0$ then $z = \partial(sz) \implies z$ is a boundary.
Define: 0th face of $s\sigma$ is $\sigma$.
$(s\sigma) \circ \delta_0 = \sigma$
$(s\sigma) \circ \delta_{j+1} = s(\sigma \circ \delta_j) \implies s\partial + \partial s = \text{id}_{S_n X} \ \forall n > 0$.
Thus if $z \in S_n X$ then ,
$s\partial z + \partial s z = z \implies \partial(sz) = z \implies z$ is a boundary. $\square$

## Algebra

**Definition.** Chain map $f_\bullet, g_\bullet : C_\bullet \to C'_\bullet$ are chian homotopic if $\exists$ homomorphisms $h_n : C_n \to C'_{n+1}$ 'degree one' such that:

$$\partial' h + h_{n-1}\partial_n = f_n - g_n$$

$$\begin{tikzcd}
C_{n+1} \arrow[r] & C_n \arrow[r] \arrow[dl, "h"] \arrow[d, "f-g"] & C_{n-1} \arrow[dl, "h"] \\
C'_{n+1} \arrow[r] & C'_n \arrow[r] & C'_{n-1}
\end{tikzcd}$$

# Wednesday, 1/29/2025

We write it as $f_\bullet \underset{h}{\simeq} g$ if $\partial'_{n+1}h_n + h_{n-1}\partial_n = f_n - g_n$. Then $f_\bullet$ and $g_\bullet$ are chain homotopic.
Homotopy theorem: $f \simeq g : X \to Y \implies f_* = g_* : H_*X \to H_*Y$
$\star$ Exercise: $f_\bullet \simeq g_\bullet \implies H_n(f_\bullet) = H_n(g_\bullet) : H_n(C_\bullet) \to H_n(C'_\bullet)$.
Hint: $(f_n - g_n)(\text{cycle}) =\in$ boundary
$\star\star$ $H_n(\Delta^n \times I) = 0$ for all $n > 0$.
$\Delta^n \times I \subset \mathbb{R}^{n+1}$
convex $\implies$ star shaped.
Homotopy: $\text{id} : X \times I \to X \times I$
Let $\eta_1 \simeq \eta_0 : X \to X \times I$
$\eta_0(x) = (x, 0)$
$\eta_1(x) = (x, 1)$
What we want to show that our theorem is true in this case.

**Lemma 16.** $\exists$ homomorphism $P_n^X : S_nX \to S_{n+1}(X \times I)$, natural in $X$, such that $\partial P + P\partial = S(\eta_1) - S(\eta_0)$.

*Proof.* Recall that $S_n : \text{TOP} \to \text{Ab}, \eta_0 : X \to X \times I \implies S_n(\eta_0) : S_nX \to S_nX \times I =: \eta_{0\#}$ is given by composition:

$$\eta_{0\#}\left(\sum_i n_i\sigma_i\right) = \sum_i n_i(\eta_0 \circ \sigma_i)$$

We prove by induction on $n$.
$N_n$ by 'naturality': $X \xrightarrow{f} Y$ and we have commutative diagram:

$$\begin{tikzcd}
S_nX \arrow[r] \arrow[d] & S_nY \arrow[d] \\
S_{n+1}(X \times I) \arrow[r] & S_{n+1}(Y \times I)
\end{tikzcd}$$

$(H_n) :\ \partial_{n+1}P_n^X + P_{n-1}^X\partial_n = S_n(\eta_1) - S_n(\eta_0)$.
$n = 0$: $0 = P_{-1} : 0 \to S_0(X \times I)$.
$P_0(\sigma)(t_0, t_1) = (\sigma(0), t_1)$
boundary of 1 chain is subtraction of midpoint.
$\partial P_0(\sigma) = \eta_1 \circ \sigma - \eta_0 \circ \sigma$
Assume $P_0^X, \cdots, P_0^X, \cdots, P_{n-1}^X$ are defined satisfying Hs and Ks.
Let $\iota = \iota_n = \text{id}_{\Delta^n}$. Goal is to define $P_n^{\Delta^n}(\iota) \in S_{n+1}(\Delta^n \times I)$.
We want $H_n$ to hold.
So we want $\partial P(\iota) = \eta_{1\#}\iota - \eta_{0\#}\iota - P\partial\iota$.
So we basically want to know whether $\eta_{1\#}\iota - \eta_{0\#}\iota - P\partial\iota$ is a cycle.
We see $\partial(\eta_{1\#}\iota - \eta_{0\#}\iota - P\delta\iota)$. Since $\eta_{i\#}$ are chain maps they commute with $\partial$ therefore:
$= \eta_{1\#}\partial\iota - \eta_{0\#}\partial\iota - \partial P\partial\iota$
$= \eta_{1\#}\partial\iota - \eta_{0\#}\partial\iota - (\eta_{1\#} - \eta_{0\#} - P\partial)(\partial i) = 0$
$\star\star \implies$ cycles = boundaries.

Choose $P_n\iota$ such that:
$\partial P_n\iota = \eta_{1\#} - \eta_{0\#} - P\partial\iota$.
Now, $(N_n)$:
$\sigma : \Delta^n \to X, \sigma_{\#} : \iota \mapsto \sigma$

$$\begin{tikzcd}
S_n\Delta^n \arrow[r, "\sigma\#"] \arrow[d, "P"] & S_nX \arrow[d] \\
S_{n+1}(\Delta^n \times I) \arrow[r, "(\sigma\times id)_{\#}"] & S_{n+1}(X \times I)
\end{tikzcd}$$

Define $P\sigma = (\sigma \times \mathrm{id})_{\#}P\iota$

$\square$

**Theorem 17** (Homotopy Theorem)**.** $f \simeq g : X \to Y \implies f_* = g_* : H_*X \to H_*Y$.

*Proof.* $H : X \times I \to Y$. We want $h_n : S_nX \to S_{n+1}Y$ such that:

$$\partial h + h\partial = H_{1\#} - H_{0\#}$$

Note: $H_i = H \circ \eta_i$
We define $h_n\sigma \coloneqq H_{\#}(P_n\sigma)$
$(\partial h + h\partial)\sigma = \partial H_{\#}P\sigma + H_{\#}P_n\partial\sigma = H_{\#}(\partial P\sigma + P\partial\sigma) = H_{\#}(\eta_{1\#} - \eta_{0\#})\sigma = H_{1\#}\sigma - H_{0\#}\sigma$ $\square$

# Friday, 1/31/2025

Exact sequences
Long Exact Sequences (LES)
Short Exact Sequences (SES)
Mayer-Vietoris Exact Sequences (MVES)
$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is <u>exact</u> if $\operatorname{im}\alpha = \ker\beta$
$\iff \operatorname{im}\alpha \subset \ker\beta$ and $\operatorname{im}\alpha \supset \ker\beta$
Note that $\operatorname{im}\alpha \subset \ker\beta \iff \beta\circ\alpha = 0$
A sequence $C_n \to C_{n-1} \to \cdots C_1 \to 0$ if it is exact at $C_{n-1}, \cdots, C_1$.
Sequence,

$$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$$

is exact if it is exact at $C_i$ for all $i$. LES.
$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$ is exact $\iff$ $C_\bullet$ is a chain complex and $H_*C_\bullet = 0$.
$0 \to A \xrightarrow{\alpha} B$ is exact $\iff \ker\alpha = 0 \iff \alpha$ is injective / 1-1.
Dual: $A \xrightarrow{\alpha} B \to 0$ is exact $\iff \operatorname{im}\alpha = B \iff \alpha$ is surjective / onto.
$0 \to A \xrightarrow{\alpha} B \to 0$ exact $\iff \alpha$ is 1-1 and onto $\iff \alpha$ is an isomorphism.
If $A \hookrightarrow B$ then $0 \to A \to B \to B/A \to 0$ is exact.
If $A \xrightarrow{f} B$ then $0 \to \ker f \to A \xrightarrow{f} B \to \operatorname{coker} f \to 0$.
Short Exact Sequence:
Suppose $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$
$\iff \alpha$ 1-1, $\beta$ onto, $\operatorname{im}\alpha = \ker\beta \iff \alpha$ 1-1, $\overline{\beta} : B/\operatorname{im}\alpha \xrightarrow{\cong} C \to 0$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\cong"] & B \arrow[r, "\beta"] \arrow[d, "="] & C \arrow[r] \arrow[d, "(\overline{\beta})^{-1}", "\cong"'] & 0 \\
0 \arrow[r] & \alpha(A) \arrow[r] & B \arrow[r] & {} \arrow[r] & 0
\end{tikzcd}$$

Canonical Example:
$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ and $0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$.
Spaces $A, B \subset X$

$$\begin{tikzcd}
A \cap B \arrow[r, hook, "i"] \arrow[d, hook, "j"] & A \arrow[d, hook, "k"] \\
B \arrow[r, hook, "\ell"] & X
\end{tikzcd}$$

**Theorem 18** (Mayer-Vietoris Exact Sequence)**.** If $X = \operatorname{int} A \cup \operatorname{int} B$ [eg $X = A \cup B, A, B$ open]
Then $\exists$ LES:

$$\cdots \to H_n A \cap B \xrightarrow{i_* \oplus j_*} H_n A \oplus H_n B \xrightarrow{k_* - \ell_*} H_n X \xrightarrow{\partial} H_{n-1} A \cap B$$

$$\cdots \to H_0 X \to 0$$

We need $\partial[\alpha]$. If $\alpha$ is a cycle in $X$ we have,

$$\alpha \underset{\text{homologous}}{\sim} \alpha_A + \alpha_B$$

Meaning $\alpha - (\alpha_A + \alpha_B)$ is a boundary.
Furthermore, $\partial[\alpha] = \partial[\alpha_A]$.
Homology of a circle $S^1$.
Circle can be written as union of $A = \cup$ and $B = \cap$.
Then $A \simeq pt, B \simeq pt, A \cap B \simeq 2pts$ which is $S^0$.
$H_1 S^1 \xrightarrow{\partial} H_0 A \cap B \to H_0 A \oplus H_0 B \to H_0 S^1 \to 0$
$0 \to H_1 S^1 \to \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{Z}^2 \to H_0 S^1 \to 0$
Thus, $H_1 S^1 = \ker \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \mathbb{Z} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cong \mathbb{Z}$
$H_0 S^1 = \operatorname{coker} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cong \mathbb{Z}$.
$0 \to H_n S^1 \to 0 \to \implies H_n S^1 = 0$.
For $n > 0$, the homology of $S^n$ can be done similarly with 2 $D^n$.

$$H_i S^n \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

Can be proven via MVES and induction on $n$.

# Monday, 2/3/2025

**Theorem 19.** For $n > 0$, we have:

$$H_i S^n \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* MVES + induction on $n$.
For $n = 1$:
Homology of a circle $S^1$.
Circle can be written as union of $A = \cup$ and $B = \cap$.
Then $A \simeq pt, B \simeq pt, A \cap B \simeq 2pts$ which is $S^0$.
$H_1 S^1 \xrightarrow{\partial} H_0 A \cap B \to H_0 A \oplus H_0 B \to H_0 S^1 \to 0$
$0 \to H_1 S^1 \to \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{Z}^2 \to H_0 S^1 \to 0$
Thus, $H_1 S^1 = \ker \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \mathbb{Z} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cong \mathbb{Z}$
$H_0 S^1 = \operatorname{coker} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cong \mathbb{Z}$.
Thus it is indeed true for $S_1$.

For $S^n$ divide Let $N = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$ and $S \equiv \begin{bmatrix} 0 \\ \vdots \\ -1 \end{bmatrix}$. Write $A = S^n - \{S\}, B = S^n - \{B\}$.
Then, $A \simeq \{N\}, B \simeq \{S\}, A \cap B \simeq S^{n-1}$.
These are deformation retracts: One way it's an inclusion, other way it is a retract.
i. $A \simeq N$: Use normalized straight line homotopy.

$$H : A \times I \to A, x \mapsto \frac{(1-t)x + tN}{\|(1-t)x + tN\|}$$

We avoid the south pole to avoid division by 0.
Same for ii.
$A \cap B$: We don't have north and south pole.
Idea: project and normalize.
Suppose $x = (x_0, \cdots, x_n) \in A \cap B$.
Let $r(x) := \frac{(x_0, \cdots, x_{n-1}, 0)}{\|x_0, \cdots, x_{n-1}, 0\|}$
Then $H(x,t) = \frac{(1-t)+tr(x)}{\|\|}$.
We're essentially going along an arc to minimize our journey throuth the sphere.

| $i$ | $H_i A \cap B$ $\cong H_i(S^{n-1})$ | $H_i A \oplus H_i B$ $\cong H_i(pt) \oplus H_i(pt)$ | $H_i S^n$ |
|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ |
| 1 | 0 | 0 | ? |
| $\vdots$ | | | |
| $n-1$ | $\mathbb{Z}$ | 0 | ? |
| $n$ | 0 | 0 | ? |

Table 2: MVES for $S^n$

All question marks should be 0 except for the last one since it maps to $\mathbb{Z}$.
Application for $H_* S^n : S^n \simeq S^m \implies n = m$.

$\square$

**Definition.** $A \subset X$ is a retract if $\exists$ continuous $r : X \to A$ such that $r(a) = a \forall a \in A$.
$A$ is a retract. $r$ is a retraction.
Thus, retract means $r$ is a left inverse of the inclusion map $i : A \to X$.
By functoriality, the same thing is true once we pass to homology.
$r_* \circ i_* = \mathrm{id}_{H_* A} \implies H_* X = i_* H_* A \oplus \ker r_*$.

**Theorem 20** (Brouwer No Retraction Theorem). $S^{n-1}$ is <u>not</u> a retract of $D^n$.

*Proof.* We use contradiction. Assume $\exists r : D^n \to S^{n-1}$ such that $r\big|_{S^{n-1}} = \mathrm{id}$.

$$\begin{tikzcd} S^{n-1} \arrow[r, "i"] \arrow[rr, bend right, "id"'] & D^n \arrow[r, "r"] & S^{n-1} \end{tikzcd}$$

Applying the functor $H_n$,

$$\begin{tikzcd} H_n S^{n-1} \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & H_n D^n \arrow[r, "r_*"] & H_n S^{n-1} \end{tikzcd}$$

$D^n$ is retractible. When $n > 1$ we have $H_{n-1} D^n = \mathbb{Z}$.

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend right, "id"'] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Which cannot happen.

$\square$

**Theorem 21** (Brower Fixed Point Theorem)**.** Every $f : D^n \to D^n$ has a fixed point.

*Proof.* Assume there doesn't exist a fixed point.
Then we can construct a retraction map $r : D^n \to S^n$ by drawing a line from $f(x)$ to $x$ and intersecting it with boundary. $\square$

## Relative Homology

Goal: exact of a pair $A \subset X$.

$$\cdots \to H_n A \to H_n X \to H_n(X, A) \xrightarrow{\partial} H_{n-1} \to \cdots$$

$H_n(X, A)$: "relative homology". We extract 'relative cycles'.
Chain is a formal linear combination. We require the boundary be in $A$.

**Definition** (Category of Pairs)**.** Top$^2$.
Objects: $(X, A)$ where $A \subset X$ open.
Morphism: $(X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ so that $f(A) \subset B$.

We have a functor $H_n : \text{Top}^2 \to \text{Ab}$.

$$\begin{tikzcd} S_n A \arrow[r, "\partial"] \arrow[d, hook] & S_{n-1} A \arrow[d, hook] \\ S_n X \arrow[r, "\partial"] & S_{n-1} X \end{tikzcd}$$

This means we have $S_n X / S_n A \xrightarrow{\overline{\partial}} S_{n-1} X / S_{n-1} A$
Relative Chain complex:

$$S_\bullet(X, A) = \left\{ \frac{S_* X}{S_* A}, \overline{\partial} \right\}$$

# Wednesday, 2/5/2025

Suppose we have a Mayer-Vietoris Exact sequence.

$$\cdots \to H_n(A \cap B) \xrightarrow{H_n(i) \oplus H_n(j)} H_n(A) \oplus H_n(B) \to \underline{H_n X} \to H_{n-1}(A \cap B) \to \cdots$$

Then we have the following short exact sequence:

$$0 \to \text{coker}(H_n i \oplus H_n j) \to H_n X \to \ker(H_n i \oplus H_n j) \to 0$$

Then, $H_n X$ is determined upto an extension.
If $\text{coker} = \mathbb{Z}/2, \ker = \mathbb{Z}/2$ then $H_n X$ can be $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.
If all of them are free [so maps are matrix maps] then we can use smith normal form.
This gives us: $H_n S^n \cong H_{n-1} S^{n-1}$.

**Remark.** Generator of $H_n S^n$ is represented by:

$$\Delta^n \to \Delta^n / \partial \Delta^n \cong S^n$$

or,

$$\partial_{n+1}(\Delta^{n+1} \to \Delta^{n+1})$$

aka sum of top simplicies of $\partial \Delta^n$.

## Relative Homology

Consider the pair $(X, A)$ with $A \subset X$.

**Definition** (Singular Relative Chain Complex)**.**

$$S_\bullet(X, A) = \left\{S_*(X, A), \overline{\partial}\right\}$$

$$\cdots \to S_{n+1}(X, A) \xrightarrow{\overline{\partial}_{n+1}} S_n(X, A) \xrightarrow{\overline{\partial}_n} S_{n-1}(X, A) \to \cdots$$

$$S_n(X, A) \coloneqq S_nX/S_nA$$

[Quotient of free abelian group, basis is a subset. So quotient basis is the complement, quotient is free abelian]
Then it is a free abelian group with basis:

$$\left\{\sigma : \Delta^n \to X \middle| \sigma(\Delta^n) \not\subseteq A\right\}$$

$$\overline{\partial} : S_n(X, A) \to S_{n-1}(X, A)$$

induced by $\partial : S_nX \to S_{n-1}X$.
or $\overline{\partial}(c + S_nA) = \partial c + S_{n-1}A$

**Definition** (Relative Homology)**.** Relative homology is the homology of the chain complex:

$$H_n(X, A) = H_n(S_\bullet(X, A)) = \frac{\ker \overline{\partial_n}}{\operatorname{im} \overline{\partial_n}} = \frac{Z_n(X, A)}{B_n(X, A)}$$

$$\xleftarrow{\cong} \frac{\pi^{-1}(Z_n(X, A))}{\pi^{-1}(B_n(X, A))} = \frac{\{c \in S_nX \mid \partial c \in S_{n-1}A\}}{\{c \in S_nX \mid \exists d \in S_{n+1}X, \text{s.t.} \partial d - c \in S_nA\}}$$

Example: $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$
For all $n$ we have $H_n : \text{Top}^2 \to \text{Ab}$ or we can consider $H_* : \text{Top}^2 \to \text{Gr}$.
We have maps induced by morphisms: $f(X, A) \to (Y, B)$ s.t. $f(A) \subset B$.
We have a corresponding map of chain compplexes:

$$S_\bullet f \coloneqq f_\# \coloneqq S_\bullet(X, A) \to S_\bullet(y, B)$$

$$H_* f = f_* : H_*(X, A) \to H_*(Y, B)$$

It is a chain map [aka it commutes with the boundary] [see above].

## Homotopy Invariance

Suppsoe we have a homotopy $H : X \times I \to Y$ so that $H(A \times I) \subset B$.
Let $f = H_0, g = H_1$.
We write $f \simeq g$.

**Theorem 22.** $f_* = g_* : H_n(X, A) \to H_n(Y, B)$

*Proof.* Same as absolute case. □

If we want to be fancy we can say:

$$H_* : \text{hTop}^2 \to \text{Gr}$$

Objects are pairs of topolocial spaces, morphisms are homotopy classes of morphisms.

**Remark.** $(X, \varnothing) \to (X, A)$ is a map of pairs.
Clear from defiinion that $H_n(X, \varnothing) = H_nX$ so we have a map $H_nX \to H_n(X, A)$.

**Theorem 23** (LES of a pair)**.** $\exists$ LES like this:

$$\cdots \to H_n A \xrightarrow{inc} H_n X \xrightarrow{\text{induced}} H_n(X, A) \xrightarrow{\partial} H_{n-1} A \to \cdots$$

Example:

$$\underbrace{H_n D^n}_{=0} \to H_n(D^n, S^{n-1}) \to H_{n-1} S^{n-1} \to \underbrace{H_{n-1} D^n}_{=0}$$

$$\implies H_n(D^n, S^{n-1}) \cong \mathbb{Z}$$

$$H_i(D^n, S^{n-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n; \\ 0, & \text{otherwise.} \end{cases}$$

Generator $\Delta^n \to \Delta^n$.

*Proof.* We have a short exact sequence of chain complexes:

$$0 \to S_\bullet A \to S_\bullet X \to S_\bullet(X, A) \to 0$$

So, all maps are chain maps.
It is levelwise a SES of abelian groups.

$$\forall n, 0 \to S_n A \to S_n X \to S_n(X, A) \to 0$$

**Lemma 24** (Zig-Zag Lemma)**.** (Theorem 2.16 of Hatcher) Slogan: SES of chain complexes gives a long exact sequence in homology.

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$$

$$\cdots \to H_n A_\bullet \xrightarrow{i_*} H_n B_\bullet \xrightarrow{j_*} H_n C_\bullet \xrightarrow{\partial} H_{n-1} A_\bullet \to H_{n-1} B_\bullet$$

Zig-Zag lemma $\implies$ LES of pair.
Proof of Zig-Zag lemma: diagram chasing.
We need to define the boundary map $\partial$. Also, we need to prove exactness at the following: $H_n A, H_n B, H_n C$.
We have 6 inclusions $\subset\supset$. Look at Hatcher!
$\partial : H_n C \to H_{n-1} A, \partial[c]$.

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r, "b \mapsto c \text{ cycle}"] \arrow[d, "b \mapsto \partial b"] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i", "a \mapsto \partial b"'] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\partial[c] = [i^{-1} \partial j^{-1} c] \in H_{n-1} A$$

$$\partial_{zz} = i^{-1} \partial j^{-1}$$

$\square$

# Friday, 2/7/2025

**Theorem 25** (Zig-Zag Lemma)**.** A SES:

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$$

Induces a LES on homology:

$$\cdots \to H_n A_\bullet \xrightarrow{i_*} H_n B_\bullet \xrightarrow{j_*} H_n C_\bullet \xrightarrow{\partial} H_{n-1} A_0 \bullet \to \cdots$$

*Proof.* First we explicitly define the boundary map $\partial : H_n C_\bullet \to H_{n-1} A_\bullet$ given by $\partial = i^{-1} \partial^B j^{-1}$.
Suppose $c \in \ker \partial^c$. From surjectivity of $j$ we can choosee $b \in B_n$ such that $jb = c$.
$j\partial^B b = \partial^c jb = \partial^c c = 0$.
Thus, $c \in \ker \partial^c$. Then we can find $a$ such that $i(a) = \partial^B b$.
We define $\partial[c] = [a]$.
Details:
$a$ is a cycle. $i\partial^A a = \partial^B ia = \partial^B \partial^B B = 0$. $i$ is injective $\implies \partial^A a = 0$.
$[a]$ is independent of the choice of $b$. Suppose $jb = c = jb'$. Then there exists $a''$ such that $ia'' = b - b'$.
We choose $a, a'$ such that $ia = \partial^B b, ia' = \partial^B b'$. Then $a - a' = \partial^A a'' \implies [a] = [a'] \in H_{n-1} A$
$[a]$ is independent of the choice of $c$. Suppose $c - c' = \partial c''$. We can find $b''$ such that $j(b'') = c''$. Then $\partial b'' \mapsto c - c'$. Thus, $\partial[c - c'] = [0]$. Thus, $\partial[c] = \partial[c']$

□

# Monday, 2/10/2025

## Reduced Homology, Excision

Preview of Redued Homology: suppose $X$ is path connected. Then,

$$\widetilde{H_n} X = \begin{cases} H_n X, & \text{if } n > 0; \\ 0, & \text{if } n = 0 \text{ and } X \text{ is path-connected.} \end{cases}$$

**Definition** (Augmentation)**.** $\varepsilon : S_0 X \to \mathbb{Z}, \sum_i n_i \sigma_i \mapsto \sum_i n_i$

Note: $\varepsilon \circ \partial_1 = 0$
$\varepsilon(\partial_1(\sigma : \Delta^1 \to X)) = \varepsilon(\sigma(1) - \sigma(0)) = 1 - 1 = 0$

**Definition.** $\widetilde{H_n} X$ is the homplogy of the agumented chain complex:

$$\cdots \to S_2 X \to S_1 X \to S_0 X \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$n > 0$ $\widetilde{H_n} X = H_n X$
SES $0 \to \frac{\ker \varepsilon}{\operatorname{im} \partial_1} \to \frac{S_0 X}{\operatorname{im} \partial_1} \xrightarrow{\varepsilon} \mathbb{Z} \to 0$
$0 \to \widetilde{H_0} X \to H_0 X \xrightarrow{\varepsilon} \mathbb{Z} \to 0$
$\widetilde{H_0} X = \mathbb{Z}^{(\# \text{ of p.c.}) - 1}$

$X_0 \in X$
$\widetilde{H_0} X \to H_0 X \to H_0(X, x_0)$
$\widetilde{H}_0 X$ is subgroup, $H_0(X, x_0)$ is quotient. They're isomorphic.
Why bother?

i) $\widetilde{H_*}(pt) = 0$

ii) Mayer Vietoris works with $\widetilde{H}$.

$$\cdots \to \widetilde{H}_n A \cap B \to \widetilde{H}_n A \oplus \widetilde{H}_n B \to \widetilde{H}_n X \xrightarrow{\partial}$$

eg consider the $X - S^1$ case.

$$0 \to \widetilde{H}_1(S^1) \to \underbrace{\widetilde{H}_0(S^1 - N - S)}_{\cong \mathbb{Z}} \to \cdots$$

Thus $\widetilde{H}_1(S^1) \cong \mathbb{Z}$

iii) $\widetilde{H_n S^n} \cong \mathbb{Z}\ \forall n \geq 0$. In particular, $\widetilde{H}_0(\varnothing) \cong \mathbb{Z}$.

iv) $\widetilde{H_n} S^n \cong \widetilde{H_{n+1}}(S^{n+1})$. Suspension isomorphism.

v) Define $\widetilde{H_i}(X, A) = \widetilde{H_i}(X, A)$. Then we have SES:

$$\to \widetilde{H_i} A \to \widetilde{H_i} X \to \widetilde{H_i}(X, A) \to \cdots$$

vi) For "good pairs" $(X, A)$,

$$H_i(X, A) \to H_i(X/A, A/A)$$

$$H_i(X, A) \xrightarrow{\cong} \widetilde{H_i}(X/A)$$

vii) There exists a cofibration exact sequence for good pairs:

$$\to \widetilde{H_i} A \to \widetilde{H_i} X \to \widetilde{H_i}(X/A) \to \widetilde{H_{i-1}} A$$

## Excision

**Definition** (Triad)**.** A triad $(X; A, B)$ means we have a topological space $X$ and $A, B \subset X$. Then,

$$S_\bullet^{\{A,B\}} := S_\bullet A + S_\bullet B \subset S_\bullet X$$

generated by $\sigma : \Delta^n \to A$ or $\Delta^n \to B$.

**Lemma 26.** Let $(X; A, B)$ be a triad. Then TFAE:

1) $H_*(B, A \cap B) \xrightarrow{\cong} H_*(X, A)$ is an isomorphism

2) $H_*(S_\bullet A + S_\bullet B) \xrightarrow{\cong} H_* X$

**Definition.** $(X; A, B)$ is a <u>excisive</u> triad if i or/and ii holds.

**Theorem 27** (Excision Theorem)**.** $X = \operatorname{int} A \cup \operatorname{int} B \implies (X; A, B)$ is excisive triad.

*Proof of Lemma.* Sublemma $(*)$: There exists SES:

$$0 \to \frac{S_\bullet B}{S_\bullet A \cap B} \to \frac{S_\bullet X}{S_\bullet A} \to \frac{S_\bullet X}{S_\bullet A + S_\bullet B} \to 0$$

Sublemma $(**)$: for all SES

$$0 \to C'_\bullet \to C_\bullet \to C''_\bullet \to 0$$

$$H_*(C') \xrightarrow{\cong} H_*(C'') \iff H_*(C''_\bullet) = 0$$

Sublemma $*$ proof: We use Noether's isomorphism theorems. We have the following SES:

$$0 \to S_\bullet A + S_\bullet B \to S_\bullet X \to \frac{S_\bullet X}{S_\bullet A + S_\bullet B} \to 0$$

Mod out by $S_\bullet A$:

$$0 \to \frac{S_\bullet A + S_\bullet B}{S_\bullet A} \to \frac{S_\bullet X}{S_\bullet A} \to \frac{S_\bullet X}{S_\bullet A + S_\bullet B} \to 0$$

Note that $\frac{S_\bullet A + S_\bullet B}{S_\bullet A} \cong \frac{S_\bullet B}{S_\bullet A \cap B}$ so we're done.
Sublemma $(**)$ follows from the zigzag lemma.
Now onto the main proof:
(1) $\overset{(*),(**)}{\iff} H_*\left(\frac{S_\bullet X}{S_\bullet A + S_\bullet B}\right) = 0 \overset{(**)}{\iff}$ (2)

$\square$

**Proposition 28.** If $(X; A, B)$ is excisive then there exists Mayer Vietoris Exact sequence:

*Proof.* First Proof:

$$0 \to S_\bullet(A \cap B) \xrightarrow{\binom{i}{j}} S_\bullet A \oplus S_\bullet B \xrightarrow{k-l} S_\bullet A + S_\bullet B \to 0$$

Apply zigzag lemma and 2 $\implies$ Mayer Vietoris.
Second Proof:

$$\begin{tikzcd}
{} \arrow[r] & H_n(A \cap B) \arrow[r] \arrow[d] & H_n B \arrow[r] \arrow[d] & H_n(B, A \cap B) \arrow[r] \arrow[d, "\cong(2)"] & H_{n-1} A \cap B \arrow[d] \\
& H_n A \arrow[r] & H_n X \arrow[r] \arrow[urr, bend right, "\partial"'] & H_n(X, A) \arrow[r] & H_{n-1} A
\end{tikzcd}$$

$\square$

# Wednesday, 2/12/2025

We give examples of excisive triads.

$$(S^n; S^n - S, S^n - N)$$

$$(S^n; S^n_+, S^n_-)$$

$$(I; \{0\}, \varnothing)$$

Goal is to discuss a proof of the Excision Theorem: If $X = \operatorname{int} A \cup \operatorname{int} B$ then $(X; A, B)$ is an excisive triad. This implies Mayer Vietoris.

*Proof of Excision Theorem.* We have $H_*(S_\bullet A + S_\bullet B) \to H_* X$.
Is it onto? We want to check if $\gamma \in S_n X$ is homologous to $\alpha + \beta \in S_n A + S_n B$. So, we want to find $\eta$ such that $\partial \eta = \gamma - (\alpha + \beta)$. Idea: subdivide $\gamma$ to $\alpha$ and $\beta$ so the tiny pieces all lie on $A$ or $B$. We can do this by Lebesgue numbers. $\square$

We're going to gack to simplices in $v_0, \cdots, v_p \in \mathbb{R}^N$.
A geometric $n$-simplex $\Delta = \langle v_0, \cdots, v_p \rangle =$ the convex hull.
$\sum_i t_i v_i \in \Delta, 0 \leq t_i \leq 1, \sum t_i = 1$.
The center is called the barycenter $b = b_\Delta = \frac{\sum_i v_i}{p+1}$, the center of mass.
We have the barycentric subdivision $\Delta'$. This will be a collection of $p$-simplices whose union will be the whole thing.

$$\Delta' = \left\{ \langle b, w_0, \cdots, w_{p-1} \rangle \middle| \langle w_0, \cdots, w_{p-1} \rangle \in \text{barycentric subdivision of a facet of } \Delta \right\}$$

A facet of $\Delta$ is given by $\langle v_0, \cdots, \widehat{v_i}, \cdots, v_p \rangle$
We need the following theorem to prove excision theorem.

**Theorem 29** (Subdivision Theorem)**.** See tom dieck, hatcher.
$\exists$ chain $\beta_\bullet = \beta_\bullet^X : S_\bullet \to S_\bullet$, natural in $X$ such that:

1) $\beta_\bullet^X \simeq \mathrm{id}_{S_\bullet X}$ chain homotopy natural in $X$. Reason: we want to make sure homologically it doesn't change things. As a consequence of this, image of cycle is cycle.

2) $\beta_p(\sigma : \Delta^p \to X)$ is supported in $(\Delta^p)'$.
If $\beta_p(\sigma) = \sum_i n_i \sigma_i$ then $\sigma_i(\Delta^p) \subset \sigma(\tau)$ for some $\tau \in (\Delta^p)'$.

Note: natural in $X$ means if we have $f : X \to Y$ then we have a commutative diagram:

$$\begin{array}{ccc} S_\bullet X & \xrightarrow{f_\#} & S_\bullet U \\ \downarrow{\scriptstyle \beta_\bullet^X} & & \downarrow{\scriptstyle \beta_\bullet^Y} \\ S_\bullet X & \xrightarrow{f_\#} & S_\bullet Y \end{array}$$

For a chain homotopy $h_p^X : S_p X \to S_{p+1} X$ to be natural means $\partial h + h\partial = \beta - \mathrm{id}$ and we have the following commutative diagram:

$$\begin{array}{ccc} S_p X & \longrightarrow & S_p Y \\ \downarrow & & \downarrow \\ S_{p+1} X & \longrightarrow & S_{p+1} Y \end{array}$$

*Proof.* Construction of $\beta$.
Suppose we have convex $D \subset \mathbb{R}^N$ and vertices $v_0, \cdots, v_p \in D$. We have:

$$[v_0, \cdots, v_p] : \Delta^p \to \langle v_0, \cdots, v_p \rangle$$

$$\sum_i t_i e_i \mapsto \sum_i t_i v_i$$

For $v \in D$ define suspension map:

$$v \cdot : S_p D \to S_{p+1} D$$

$$v \cdot [v_0, \cdots, v_p] = [v, v_0, \cdots, v_p]$$

Let $\iota_p : \Delta^p \to \Delta^p$ be the identity.
We define $\beta_p^X$ by induction on $p$.
For $p = 0$ we have $\beta_0(\sigma) = \sigma$ [we cannot subdivide point].
We want [by naturality] $\beta_p(\sigma) = \sigma_\# \beta_p(\iota_p)$
$\beta_p(\iota_p) = b \cdot \beta_{p-1}(\delta \iota_p) \in S_p \Delta^p$
Proof of i, ii ommitted.
$\beta_1(\sigma)$ is given by formal difference between paths from $b$ to each endpoint for example. $\square$

Suppose we have a metric space $A$. Recall that $\operatorname{diam} A = \sup_{a,a'} d(a, a')$.

**Lemma 30.** $v_0, \cdots, v_p \in \mathbb{R}^N$. Then $\operatorname{diam}\langle v_0, \cdots, v_p \rangle = \max_{i,j} \|v_i - v_j\|$.

*Proof.* Suppose $a, a' \in \langle v_0, \cdots, v_p \rangle$. Write down barycentric coordinates:
$a = \sum_i t_i v_i, a' = \sum_i t_i' v_i$.

$$\|a - a'\| = \left\| \sum_i t_i(v_i) - a' \right\| = \left\| \sum_i t_i (v_i - a') \right\| \leq \sum_i t_i \|v_i - a'\|$$

$$\leq \left( \sum_i t_i \right) \max_i \|v_i - a'\| \leq \max_{i,j} \|v_i - v_j\|$$

$\square$

**Corollary 31.** $\Delta = \langle v_0, \cdots, v_p \rangle$ we have $\tau \in \Delta'$ and $\operatorname{diam} \tau \leq \frac{p}{p+1} \operatorname{diam} \Delta$.

# Friday, 2/14/2025

**Theorem 32** (Classical Excision Theorem)**.** If $Z \subset A \subset X, \overline{Z} \subset \operatorname{int} A$, then,

$$H_*(X - Z, A - Z) \xrightarrow{\cong} H_*(X, A)$$

Basically we can cut out (excise) $Z$.

*Proof.* Let $Z = X - B$. Then, $B$ and $Z$ are complements.
$\overline{Z} \subset \operatorname{int} A \subset X \iff X = \operatorname{int} A \cup \operatorname{int} B$. □

Slogan: you can excise closed sets from open sets.
Suppose we have a knot $S^1 \cong K \subset S^3$. Suppose we have the trefoil.
We can prove that the trefoil is not a circle by the fundamental group $\pi_1(S^3 - K)$ but not homology!
One can use Seifert-Van Kampen theorem to show that $\pi_1(S^3 - K) = \langle a, b \mid a^3 = b^2 \rangle \twoheadrightarrow \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$
$S_3$ non-abelian. Then, $\pi_1(S^3 - K) \ncong \mathbb{Z} = \pi_1(S^3 - S^1) \cong S^1 \times \mathring{D}^2$.
$N(K)$ = tubuler neighborhood of $K \cong S^1 \times D^2$ solid torus.
$H_*(N(K), N(K) - S^1) \xrightarrow{\cong} H_*(S^3, S^3 - K)$.
Also, $H_*(N(K), N(K) - S^1) = H_*(N(S^1), N(S^1) - S^1)$ so $S^3 - K$ is a homology circle $[H_*(S^3 - K) = H_*(S^1)]$.
[See: Massey Algebraic Topology]
Let $v_0, \cdots, v_p \in \mathbb{R}^n$. let $\Delta = \langle v_0, \cdots, v_p \rangle = \{\sum_i t_i v_i | 0 \le t_i \le 1, \sum_i t_i = 1\}$.

**Lemma 33.** Suppose $x \in \Delta$. Then, $\sup_{y \in \Delta} \|x - y\| = \max_j \|x - v_j\|$

*Proof.*

$$\|x - y\| = \left\| x - \sum_i t_i y_i \right\| = \left\| \sum_i t_i (x - y_i) \right\|$$

$$\le \sum_i t_i \|x - y_i\| \le \left\{ \sum_i t_i \right\} \left\{ \max_j \|x - y_j\| \right\} = \max_j \|x - y_j\|$$

□

Applying this twice, $\|x - y\| \le \max_{i,j} \|x_i - x_j\|$
The $j$-th face of $\Delta$ is $\delta_j \Delta = \langle v_i, \cdots, \widehat{v_j}, \cdots, v_p \rangle$.
The barycenter $b = \frac{v_0 + \cdots + v_p}{p+1}$. Like the centroid.
Barycentric subdivision of $\Delta$: It is going to be a collection of $p$-simplices that cover $\Delta$.

$$\Delta' = \left\{ \langle b, w_0, \cdots, w_{p-1} \rangle \mid \{w_0, \cdots, w_{p-1}\} \in (\partial_j \Delta)' \right\}$$

**Corollary 34.** $\tau \in \Delta' \implies \operatorname{diam}(\tau) \le \frac{p}{p+1} \operatorname{diam} \Delta$.

*Proof.* We use induction on $p$. Suppose $\tau = \langle b, w_0, \cdots, w_{p-1} \rangle$.
Case 1: $\operatorname{diam} \tau = \|w_i - w_j\|$. $\langle w_0, \cdots, w_{p-1} \rangle \in (\partial_j \Delta)' \implies \|w_i - w_j\| \le \frac{p-1}{p} \operatorname{diam} \partial_j \Delta$ by inductive hypothesis and $< \frac{p}{p+1} \operatorname{diam} \Delta$.
Case 2: Suppose $\operatorname{diam} \tau = \|b - w_i\|$.
$\|b - w_i\| \le \|b - v_j\|$ for some $j$ by the lemma.

$$\|b - v_j\| = \left\| \frac{1}{p+1} \sum_k v_k - v_j \right\| = \left\| \frac{1}{p+1} \left( \sum_k v_k - v_j \right) \right\|$$

$$\le \frac{1}{p+1} \sum_k \|v_k - v_j\| \le \frac{1}{p+1} \sum_{k \ne j} \max_l \|v_l - v_j\| = \frac{p}{p+1} \operatorname{diam} \Delta$$

□

As we subdivide we have $\lim_{k \to \infty} \left( \frac{p}{p+1} \right)^k = 0$ which is the point.

**Proposition 35.** If $X = \operatorname{int} A \cup \operatorname{int} B$ and $c \in S_pX$ is some chain then $\exists k$ such that,

$$\beta^k \subset S_pA + S_pB$$

$k$ is the subdivision operator.

This uses the Lebesgue Number Theorem [a fact about comapct metric spaces]:

**Theorem 36.** Let $\mathcal{U}$ be an open cover of a compact metric space $(A, d)$ then, $\exists \epsilon > 0$ [called the Lebesgue number] such that if we have $B \subset A$ such that $\operatorname{diam} B < \epsilon$ then $\exists O \in \mathcal{U}$ such that $B \subset O$.

Using the theorem we can prove the proposition.

*Proof of Proposition.* It suffices to prove this when $c = \sigma : \Delta^p \to X$. We choose a Lebesgue number $\epsilon$ for the open cover $\{\sigma^{-1} \operatorname{int} A, \sigma^{-1} \operatorname{int} B\}$.
Choose $k$ such that $\left(\frac{p}{p+1}\right)^k < \epsilon$.
Part ii of the subdivision theorem implies $\operatorname{supp}(\beta^k\sigma)$ is contained in $(\Delta^p)^k$.
i.e. $\beta^k\sigma = \sum_i n_i\sigma_i$.
$\sigma_i(\Delta^p) \subset \sigma(\tau)$ for some $\tau \in (\Delta^p)^k$.
$\operatorname{diam} \tau \leq \left(\frac{p}{p+1}\right)^k \epsilon$
Therefoore, $\tau \subset \sigma^{-1} \operatorname{int} A$ or $\sigma^{-1} \operatorname{int} B$.
Therefore, $\sigma(\tau) \subset \operatorname{int} A$ or $\sigma(\tau) \subset \operatorname{int} B$.

□

# Monday, 2/17/2025

Recall subdivisin theorem.

**Theorem 37** (Subdivision Theorem)**.** There exists chain map $\beta^X_\bullet \simeq \mathrm{id} : S_\bullet X \to S_\bullet X$ natural in $X$ and supported in barycentric subdivision.

**Proposition 38.** Let $X = \operatorname{int} A \cup \operatorname{int} B$. For chain $c \in S_pX, \exists k$ such that if we subdivide $k$ times aka take $\beta^k c$, we have:

$$\beta^k c \in S_pA + S_pB$$

We want to prove the following theorem.

**Theorem 39** (Excision Theorem)**.** Suppose $X = \operatorname{int} A \cup \operatorname{int} B$. Then,

$$H_*(S_\bullet A + S_\bullet B) \xrightarrow{\cong} H_*X$$

Chain homotopic implies we have equation:

$$\beta^X_\bullet - \mathrm{id} = \partial h^X + h^X \partial$$

Natural means if we hafe $f : X \to Y$ we have the following commutative square: $f_\# \circ \beta^X = \beta^Y \circ f_\#, f_* \circ h^X = h^Y \circ f_\#$.
Claim: $\beta^k_\bullet \simeq \mathrm{id}$.
Proof: $\beta \simeq \mathrm{id}, f \simeq g \implies f \circ h \simeq g \circ h (\implies \beta^i \simeq \beta^{i+1})$. $\simeq$ is an equivalence relation.

*Proof of Excision Theorem.* Onto: Let $[\gamma] \in H_pX$ be a cycle. Then $\exists k$ such that $\beta^k\gamma \in S_pA + S_pB$.
$\beta^k - \mathrm{id} = \partial h + h\partial$.
$\beta^k\gamma - \gamma = \partial h\gamma + \cancel{h\partial\gamma} \implies \gamma = \beta^k\gamma + \partial(h\gamma) \implies [\gamma] = [\beta^k\gamma] \in H_pX$.
$[\beta^k\gamma] \in \operatorname{im} H_p(S_\bullet A + S_\bullet B)$.
1-1: Suppose $[\gamma] \in \ker(H_p(S_\bullet A + S_\bullet B) \to H_pX)$. Then we can write $\gamma = \partial\eta$. There exists $k$ such that $\beta^k\eta \in S_{p+1}A + S_{p+1}B$.
$\beta^k\eta - \eta = \partial h\eta + h\partial\eta$. Apply $\partial$.

$$\partial\beta^k\eta - \partial\eta = \partial h\partial\eta$$

$$\partial\beta^k\eta - \gamma = \partial h\gamma$$

Similarly, $\gamma = \partial(\beta^k\eta - h\gamma) \in \partial(S_{p+1}A + S_{p+1}B)$
SInce $h$ is natural, $\gamma = \alpha + \beta, h\gamma = h\alpha + h\beta \in S_{p+1}A + S_{p+1}B, [\gamma] = 0 \in H_p(S_\bullet A + S_\bullet B)$. $\square$

**Remark.** Let $\mathcal{U}$ be a family of subsets from $X$ such that $X = \bigcup_{A_i\in\mathcal{U}} \operatorname{int} A_i$ and let $S^{\mathcal{U}}X = \sum S_\bullet A_i$
Proof of excision theorem shows that:

$$H_*(S_\bullet^{\mathcal{U}}X) \xrightarrow{\cong} H_*X$$

**Lemma 40** (Five Lemma)**.** If we have a commutative diagram with exact rows [like the following:]

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

If $\alpha, \beta, \delta$ and $\epsilon$ are isomorphisms, then so is $\gamma$.

*Proof.* $\gamma$ injective: Suppose $\gamma(c) = 0$.
$a \mapsto b, b \mapsto c$. $a' \mapsto b', b' \mapsto 0$. Thus, $c = 0$.
$\gamma$ surjective: Suppose $c' \in D'$ so that $c' \mapsto d' \mapsto 0$ then we have $d \mapsto 0$, preimage gives us $c \mapsto d \mapsto 0$. Since $b' \mapsto c' - \operatorname{im} c \mapsto 0$, we are going to adjust $c$ by the image of $b$.
$c + \operatorname{im} b \mapsto \operatorname{im} c + c' - \operatorname{im} c = c'$. $\square$

Typical application (2/3): let isomorphisms $f : (X, A) \to (Y, B)$ with $f_* : H_*X \to H_*Y$ and $f_* : H_*A \to H_*B$. Then, $f_* : H_*(X, A) \to H_*(Y, B)$.
Proof:

$$\begin{tikzcd}
\cdots \arrow[r] & H_nA \arrow[r] \arrow[d, "\cong"] & H_nX \arrow[r] \arrow[d, "\cong"] & H_n(X,A) \arrow[r] \arrow[d] & H_{n-1}A \arrow[r] \arrow[d, "\cong"] & H_{n-1}X \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_nB \arrow[r] & H_nY \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}B \arrow[r] & H_{n-1}Y \arrow[r] & \cdots
\end{tikzcd}$$

# Quotient Topology

Let $X$ be topological space, $\sim$ an equivalence relation. Then $X/\sim$ is the set of equivalence classes with $\pi : X \to X/\sim$.

**Definition.** $U \subset X/\sim$ is open in $X/\sim \iff \pi^{-1}(U)$ is open in $X$.

# Wednesday, 2/19/2025

We can rephrase this.

**Definition** (Saturated)**.** $V \subset X$ is <u>saturated</u> if $V$ is a union of equivalence classes, i.e. if we have $v \in V, v' \sim v \implies v' \in V$.

Then, open sets in $X/\sim$ are the image of saturated open sets in $X$.

**Definition.** $q : X \to Y$ is a <u>quotient map</u> if $q$ is continuous, $q$ is onto and $U$ is open in $Y \iff q^{-1}U$ is open in $X$.

Idea: Quotient Topology = Quuotient Map. $X \to X/\sim$ is a quotient map! Further, if $q : X \to Y$ is a quotient map we can define an equivalence relation on $X$: $x \sim x'$ if $q(x) = q(x')$. Then, $Y \cong X/\sim_q$. So the equivalence classes are $q^{-1}y$.

## Universal Property of Quotient Topology

Suppose we have a continuous $f : X \to Z$ that is constant on equivalence class. Then it factors through the quotient:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \\ X/\sim \arrow[ur, dotted, "\overline{f}"] & \end{tikzcd}$$

**Theorem 41** (Universal Property of Quotient Map)**.** Given the following Commutative Diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"] & Z \\ Y \arrow[ur, "h"] & \end{tikzcd}$$

such that $f$ is continuous and $q$ is quotient map, then $h$ is continuous.

## Recognition of Quotient Map

$q : X \to Y$ is continuous, onto.
$q$ open map $\implies$ $q$ quotient map.
$q$(open) are open.
$q$ closed map $\implies$ $q$ qm.
$q$(closed) are closed.

## Jim's Favorite Trick

If $f : X \to Y$ is continuous, $X$ compact and $Y$ Haussdorff then if $X$ is $\begin{Bmatrix} \text{onto} \\ \text{bijective} \end{Bmatrix}$ then $f$ is a $\begin{Bmatrix} \text{quotient map} \\ \text{homeomrophism} \end{Bmatrix}$

*Proof.* Idea: Closed subsets of compact spaces are compact.
Idea: A compact subset of a Hausdorff space is closed.
Then $f : X \to Y$ is closed. □

Idea: if we have $A \subset X$ then we can create $X/A := X/\sim_A$:
$a \sim_A a' \iff a, a' \in A$.
This isn't an equivalence relation, this is just a relation. There is an equivalence relation: we need to add the condition $x \sim_A x$.
Then, open sets in $X/A$ are unions of open sets on $X$ disjoint from $A$ and open sets in $X$ containing $A$.
Claim: $D^n/S^{n-1} \cong S^n$.
Consider the map $D^n \to \mathbb{R}^n \cup \{\infty\} \xrightarrow[\cong]{t} S^n$ by stereographic projection, $x \mapsto \frac{1}{1-|x|}x$

$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & S^n \\ D^n/S^{n-1} & \end{tikzcd}$$

is a quotient map by JFT, continuous bijection, hence homeomorphism by JFT.

**Definition.** $(X, A)$ is a good pair if:

$$H_*(X, A) \xrightarrow{\approx} H_*(X/A, A/A)$$

Note: $H_*(X/A, A/A) \xrightarrow{\approx} \widetilde{H}_*(X/A)$.
Then this is the same as:

$$H_*(X, A) \xrightarrow{\approx} \widetilde{H}_*(X/A)$$

eg $(D^n, S^{n-1})$ is a good pair.
$(X, A)$ is a good pair $\implies$ there is a long exact sequence:

$$\widetilde{H}_n A \to \widetilde{H}_n X \to \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1} A \to \widetilde{H}_{n-1} X$$

**Theorem 42.** If $A \subset V \subset X$, $A$ closed, $A \subset \operatorname{int} V$ and $A \to V$ is a deformation retract then $(X, A)$ is a good pair.

eg $(D^n, S^{n-1})$ is a good pair.
The proof requires exicision and homotopy invariance and the five lemma.

# Friday, 2/21/2025

Note: the previous theorem is Hatcher's definition of a good pair. These concepts are not equivalent!

*Proof.* Consider the following diagram:

$$\begin{tikzcd}
H_*(X,A) \arrow[r, "1"] \arrow[d] & H_*(X,V) & H_*(X-A, V-A) \arrow[l, "2"'] \arrow[d, "3"] \\
H_*(X/A, A/A) \arrow[r, "5"] & H_*(X/A, V/A) & H_*(X/A - A/A, V/A - A/A) \arrow[l, "4"']
\end{tikzcd}$$

We want to prove that each of them is an isomorphism.
2 is isomorphism by excision.
3 is isomorphism since we removed $A$ and then modded out $A$, so they're in fact the same spaces.
4 is isomorphism by excision.
1 and 5 are isomorphisms by homotopy invariance of homology and five lemma (cf HW ex 2). We have something like this:

$$\begin{tikzcd}
H_n A \arrow[r] \arrow[d, "\cong"] & H_n X \arrow[r] \arrow[d, "\cong"] & H_n(X,A) \arrow[r] \arrow[d] & H_{n-1} A \arrow[r] \arrow[d, "\cong"] & H_{n-1} X \arrow[d, "\cong"] \\
H_n V \arrow[r] & H_n X \arrow[r] & H_n(X,V) \arrow[r] & H_{n-1} V n \arrow[r] & H_{n-1} X
\end{tikzcd}$$

□

## Degree

We can talk about the degree of a map $f : S^n \to S^n$. The degree is an integer. Think about winding number.
$f : S^n \to S^n$ induces $H_n S^n \to H_n S^n$. Easier to consider $\widetilde{H}_n S^n \to \widetilde{H}_n S^n$. These are infinite cyclic groups so the map is a multiplication, thet number is $\deg f$.

$$\widetilde{H}_n S^n \xrightarrow{\cdot(\deg f)} \widetilde{H}_n S^n$$

We have the following property:

1) $\deg \mathrm{id} = 1$.

2) $\deg(f \circ g) = (\deg f)(\deg g)$.

3) Due to Hopf: $\deg f = \pm 1 \iff f$ is homotopy equivalence.

4) $f$ not onto $\implies \deg f = 0$.

5) $\deg(f : \widetilde{H}_0(S^0) \to \widetilde{H}_0(S^0)) \in \{-1, 0, 1\}$.

6) Consider the antipodal map $A = -\operatorname{id} : S^n \to S^n$ given by $A(x) = -x$. $\deg(-\operatorname{id}) = (-1)^{n+1} = \det(-I_{n+1})$

7) $f : S^n \to S^n$ with no fixed point then $\deg f = (-1)^{n+1}$ [HW]

8) $f : S^n \to S^n, f(x) \neq -x \forall x \implies \deg f = 1$.

9) global degree = local degree, $\deg f = \# f^{-1}(x)$.

10) $\deg(z \to z^n) = n, S^1 \to S^1$.

11) $M \in O(n) \implies \deg M = \det M$.

*Proof.* 1 and 2 follow from functoriality: $\widetilde{H}_n : \mathrm{Top} \to \mathrm{Ab}$ is a functor.
3: $\Longleftarrow$ : if $f$ is homomotpy equivalence then there is a homotopy inverse $g$, then $(\deg f)(\deg g) = 1 \implies \deg f = \pm 1$. $\implies$ is a deep theorem of Hopf. We will not prove it in this class.
4: We need lemma.

**Lemma 43.** If $f : S^n \to S^n, x_0 \notin f(S^n)$ then $f \simeq \mathrm{const}_{-x_0}$: we'll take the straight line homotopy:

$$\frac{tf(x) + (1-t)(-x_0)}{\|tf(x) + (1-t)(-x_0)\|}$$

Thus $\deg f = \deg(c_{-x_0})$.

$$\begin{tikzcd} S^n \arrow[rr, "c_{-x_0}"] \arrow[dr] & & S^n \\ & \{-x_0\} \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} \widetilde{H}_n S^n \arrow[rr] \arrow[dr] & & \widetilde{H}_n S^n \\ & \widetilde{H}_n(pt) = 0 \arrow[ur] & \end{tikzcd}$$

Recall $H_0 S^0$ has basis $\left[\sigma_1 : \underset{e_0}{\Delta^0} \underset{\mapsto}{\longrightarrow} \underset{1}{S^0}\right], \left[\sigma_{-1} : \underset{e_0}{\Delta^0} \underset{\mapsto}{\longrightarrow} \underset{-1}{S^0}\right]$
$\widetilde{H}_0 S^0$ then has basis the kernel, which is $[\sigma_1] - [\sigma_{-1}]$.
$\deg(\mathrm{id} : S^0 \to S^0) = 1, \deg(\mathrm{const} : S^0 \to S^0) = 0$.
$\deg A = ? A_*([\sigma_1] - [\sigma_{-1}]) = [\sigma_{-1}] - [\sigma_1], \deg A = -1$.

**Lemma 44.** If $f : S^n \to S^n$ has no fixed points then $f \simeq -\operatorname{id} = A$.
Once again consider the normalized straight line homotopy:

$$\frac{tf(x) + (1-t)(-x)}{\|tf(x) + (1-t)(-x)\|}$$

□

## Monday, 2/24/2025

Recall Antipodal Map $A = -\operatorname{id} : S^n \to S^n$.

**Proposition 45.** $\deg(A) = (-1)^{n+1}$.

*Proof.* We can write $A$ as a composition of reflection through hyperspheres:

$$A = R_0 \circ \cdots \circ R_n$$

Where $R_i$ flips the $i$'th coordinate:

$$R_i(x_1, \cdots, x_i, \cdots, x_n) = (x_0, \cdots, -x_i, \cdots, x_n)$$

It is a reflection throuh hypersphere.
These reflections have the same degree since they're conjugates. Let $h_i$ be the swap of $x_0$ and $x_i$. Then, $R_i = h_i \circ R_0 \circ h_i^{-1}$.
Then $\deg(R_i) = \deg(h_i)\deg(R_0)\deg(h_i^{-1}) = \deg(R_0)$.
It is enough to prove $\deg(R_0) = -1$.
We use induction to prove this.
$n = 0$ there's nothing to prove. Assume true for $n-1$.
MVES is natural, ie if we have $f : (X; A, B) \to (X', A', B'), f(A) \subset A'$ of excisive triads, then:
Thus there exists a commutative ladder of MVES:

$$\begin{tikzcd}
\cdots \arrow[r] & H_i X \arrow[r, "\partial"] \arrow[d] & H_{i-1}(A \cap B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_i X' \arrow[r, "\partial'"] & H_i(A' \cap B') \arrow[r] & \cdots
\end{tikzcd}$$

Consider:

$$R_0 : (S^n; S^n - \{e_0\}, S^n - \{-e_0\}) \to (S^n; S^n - \{-e_0\}, S^n - \{e_0\})$$

Thus we get:

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[r, "\cong"] \arrow[d, "R_{0_*}=-1"] & \widetilde{H}_{n-1}(S^n - \{e_0, -e_0\}) \arrow[r, "\cong"] \arrow[d, "R_{0_*}=-1"] & \widetilde{H}_{n-1}(S^{n-1}) \arrow[d, "R_{0_*}=-1"] \\
\widetilde{H}_n(S^n) \arrow[r, "\cong"] & \widetilde{H}_{n-1}(S^{n-1} - \{e_0, -e_0\}) \arrow[r, "\cong"] & \widetilde{H}_{n-1}(S^{n-1})
\end{tikzcd}$$

□

**Remark.** Suppose $M \in O(n)$ ie $M^T = M^{-1}$. Then, $M$ is also a composite of reflection through hyperplanes thus $\deg M = \det M$.

## Hairy Ball Theorem

Slogans: Can't comb the hairy ball
Every tangent vector field on $S^2$ has a zero
Every vector field on $S^2$ has a normal direction.
We make them precise.
Last one: suppose we have smooth $v : S^2 \to \mathbb{R}^3$. Then $\exists p \in S^2$ such that $v(p) \in \mathbb{R}p$.
We call this a 'cowlick'.

**Definition.** A tangent vector field on $S^2$ is $v : S^2 \to \mathbb{R}^3$ such that $v(x) \cdot x = 0 \, \forall x \in S^2$.

Euler Characteristic $\chi(X) = \sum_i (-1)^i \operatorname{rank}(H_i X)$.
If $M$ is a closed manifold, can comb hairy $M \iff \chi(M) = 0$.
Will not prove this now.

*Proof of Hairy Ball Theorem.* First Proof: We use contradiction.
Let $v : S^2 \to \mathbb{R}^3$ be a tangent vector field. Then $x \perp v(x)$ for all $x \in S^2$. FTSOC assume $v(x) \neq 0$. Then we can normalize:

$$f(x) = \frac{v(x)}{\|v(x)\|}$$

Then $f : S^2 \to S^2$ with $x \cdot f(x) = 0 \forall x$.

Thus, $\forall x \in S^2, f(x) \neq x, f(x) \neq -x$.
Thus we have straight line homotopies: $f \simeq A, f \simeq \text{id}$.
Thus, $A \simeq \text{id} \implies 1 = -1$.
Second Proof: We flow along the tangent vector. That gives us $\text{id} \simeq A$.

$\square$

# Wednesday, 2/26/2025

We compute the degree of $f_n : S^1 \to S^1$ where $f_n : z \mapsto z^n$.
Then, $f_n^{-1}(1) = \left\{e^{2\pi i k/n}\right\}$, the $n$-th roots of 1.

**Theorem 46.** $\deg f_n = n$.

Local degree is counting the number of inverse images. So, this is essentially saying: local degree is global degree, kinda.
$H_1 S^1$ is generated by a $\sigma : \Delta^1 \to S^1$. Then $\sigma(t_0, t_1) = e^{2\pi i t_0}$.

*Proof.* First Proof: We 'chop up' the circle into pieces, $\sigma_1, \sigma_2, \cdots, \sigma_n$. Then, $f(\sigma_j) = \sigma$.
$\sigma_j(t_0, t_1) = e^{\frac{2\pi i (j + t_0)}{n}}$.
Claim: $\sigma$ is homologous to $\sigma_1 + \cdots + \sigma_n$.
ie $\exists \gamma$ such that $\partial \gamma = \sigma - (\sigma_1 + \cdots + \sigma_n)$.
[Not gonna prove this]
Then, $(f_n)_*[\sigma] = (f_n)_*[\sigma_1 + \cdots + \sigma_n] = [f_{n\#}\sigma_1 + \cdots + f_{n\#}\sigma_n] = [\sigma] + \cdots + [\sigma] = n[\sigma]$
Second Proof: recall that $\pi_1(S^1, 1) = [(I, \{0, 1\}), (S^1, \{1\})]$.
$\pi_1(S^1, 1) \cong \mathbb{Z}$.
For any $I \xrightarrow{\alpha} S^1$ we have a lift $I \xrightarrow{\widetilde{\alpha}} \mathbb{R}$ where $\mathbb{R} \to S^1$ is $e^{2\pi i t}$, then,
$\pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}$ is given by $[\alpha] \mapsto \widetilde{\alpha}(1) - \widetilde{\alpha}(0)$.
Then, $\widetilde{f_n \circ \alpha}(1) - \widetilde{f_n \circ \alpha}(0) = n(\widetilde{\alpha}(1) - \widetilde{\alpha}(0)) \implies \pi_1(f_n) = \cdot n$
We have something called a <u>Hurewicz Map</u> $\pi_1(S^1, 1) \to H_1(S^1)$ given by $[\alpha : I \to S^1] \mapsto [\alpha : \Delta^1 \to S^1]$.

$$\begin{tikzcd}
\pi_1(S^1,1) \arrow[r, "\cdot n"] \arrow[d, "\cong"] & \pi_1(S^1,1) \arrow[d, "\cong"] \\
H_1 S^1 \arrow[r, "\implies \cdot n"] & H_1 S^1
\end{tikzcd}$$

We recall suspension. Recall: $\Sigma : \text{Top} \to \text{Top}$ is a functor.
$\Sigma X = \frac{X \times I}{(x,0) \sim (x',0), (x,1) \sim (x',1)}$.
It is a functor, so if we have map $f : X \to Y$ we have suspension of a map $\Sigma f : \Sigma X \to \Sigma Y$
Given by $[x, t] \mapsto [f(x), t]$.
MVES shows:

$$\begin{tikzcd}
\widetilde{H}_i(\Sigma X) \arrow[r, "\cong"] \arrow[d, "(\Sigma f)_*"] & \widetilde{H}_{i-1}(X) \arrow[d, "f_*"] \\
\widetilde{H}_i(\Sigma Y) \arrow[r, "\cong"] & \widetilde{H}_{i-1}(Y)
\end{tikzcd}$$

Application: we have maps of arbitrary degrees. $f_n : S^1 \to S^1$ has $\deg f_n = n$. Suspension $\implies \Sigma f_n : S^2 \to S^2$ has degree $n$, repeated suspension implies we can find map $f : S^k \to S^k$ with degree $n$.
Third Proof: global degree = local degree:

$$\deg f_n = \sum_{x \in f^{-1}(1)} \deg f_n \Big|_x = 1 + \cdots + 1 = n$$

Suppose we have $f : S^n \to S^n$, $y$ in the image. We want to define degree of $y$.
$y \in S^n$ is <u>finite value</u> of $f$ if $|f^{-1}(y)| < \infty$.

Suppose $f^{-1}y = \{x_1, \cdots, x_m\}$.
$\exists$ open $V \ni y$ and open $U_1, \cdots, U_m$ containing $x_1, \cdots, x_m$ respectively, each open disjoint such that $f(U_i) \subset V$.
In this situation, we define the local degree at $x_i$:
$H_n(U_i, U_i - x_i) \xrightarrow{f_*} H_n(V, V - y)$.
By hw, this is $\mathbb{Z} \xrightarrow{f_*} \mathbb{Z}$ and we define $\deg f$ such that this map is $\cdot \deg f$.
Theorem: if $y \in S^n$ is finite valued then $\deg f = \sum \deg f\big|_{x_i}$

$\square$

## Friday, 2/28/2025

Deviating from Hatcher, we 'redefine' local degree.
Consider $f : S^n \to S^n$. Let $x \in S^n, y = f(x)$ such that $x$ is an isolated point of $f^{-1}(y)$. Meaning, there exists a neighborhood of $U$ of $x$ such that $U \cap f^{-1}y = x$. This is equivalento to saying there exists a neighborhood $V$ of $y = f(x)$ and a neighborhood $U$ of $x$ such that $f(U) \subset V$.
This is a generalization of a finite point.
If $x$ is an isolated point of $f$, will local degree of $f$ at $x$: $\deg f\big|_x \in \mathbb{Z}$.
Idea:

$$\begin{tikzcd}
H_n(U, U - x) \arrow[r, "f_*"] \arrow[d, "\approx"] & H_n(V, V - y) \arrow[d, "\approx"] \\
\mathbb{Z} \arrow[r, "\deg f|_x"] & \mathbb{Z}
\end{tikzcd}$$

Issue: we want the $\mathbb{Z}$ to be the 'same'.
Choose generator $[S^n] \in H_n S^n \iff$ choose isomorphism $H_n S^n \cong \mathbb{Z}$: Orientation.

$$\begin{tikzcd}
H_n(U, U - x) \arrow[r, "f_*"] \arrow[d, "\cong"] & H_n(V, V - y) \arrow[d, "\cong"] \\
H_n(S^n, S^n - x) & H_n(S^n, S^n - y) \\
H_n S^n \arrow[u, "\cong"'] \arrow[d, "\approx"] & H_n(S^n) \arrow[u, "\cong"'] \arrow[d, "\approx"] \\
\mathbb{Z} \arrow[r, "\cdot \deg f|_x"] & \mathbb{Z}
\end{tikzcd}$$

Remark: this is independent of choice of $U$ and $V$.
Also: $f : V \to V$ is homeomorphism, then $\deg f\big|_X = \pm 1$.

**Theorem 47** (Global Degree = Local Degree)**.** Le $f : S^n \to S^n$. If $y \in S^n$ such that $f^{-1}y$ is a finite set, then the global degree is sum of local degree:

$$\deg f = \sum_{x \in f^{-1}y} \deg f\Bigg|_x$$

*Proof.* Suppose $f^{-1}y = \{x_1, \cdots, x_m\}$. Choose an open neighborhood $V \ni y$ and disjoint open $U_i \nu x_i$.

$$
\begin{tikzcd}
 & \mathbb{Z}^m \arrow[r] \arrow[d, "\cong"] & \mathbb{Z} \arrow[d, "\cong"] \\
 & \bigoplus_i H_n(U_i, U_i - X) \arrow[r] \arrow[d, "\cong"] \arrow[dl, "\cong"'] & H_n(V, V-y) \arrow[d, "\cong"] \\
\mathbb{Z}^m = \bigoplus_i H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}y) \arrow[l] \arrow[r] & H_n(S^n, S^n - y) \\
 & H_nS^n \arrow[u] \arrow[r, "f_*"] \arrow[ul] \arrow[d, "\cong"] & H_nS^n \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
 & \mathbb{Z} \arrow[r, "\cdot \deg f"] & \mathbb{Z}
\end{tikzcd}
$$

In the bottom $\mathbb{Z}$ we have $1 \mapsto \deg f$. But we can go all the way around the commutative diagram, $1 \mapsto \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \mapsto \sum_i \deg f\big|_{x_i}$, thus they are indeed the same.

$\square$

Example of map where local degree is 2: $\deg \Sigma(z \to z^2)\big|_N = 2$.

**Definition.** $y \in S^n$ is a regular value of $f : S^n \to S^n$ if '$f$ is a local homeomorphism near $f^{-1}y$', ie $U_i \xrightarrow{\approx} V \forall i$.

Fact: If $f : S^n \to S^n$ is smooth and if $f^{-1}y = \{x_1, \cdots, x_m\}$ and $df_{x_i} T_{x_i} S^n \to T_y S^n$ is onto, then $y$ is a regular value at $f$.

**Theorem 48** (Sard's Theorem)**.** If $f : S^n \to S^n$ is smooth then almost all $y \in S^n$ are regular values.

Suppose we have a category $\mathcal{C}$, objects $X \in \mathcal{C}$ and morphisms $\mathcal{C}(X, Y)$.

**Definition.** A product of two objects $X_1, X_2 \in \mathcal{C}$ is a triple $(X, X \xrightarrow{\pi_1} X_1, X \xrightarrow{\pi_2} X_2)$ such that it is final in the class of triples. Meaning:

$$
\begin{tikzcd}
 & & X \\
Y \arrow[r, dotted, "\exists!"] \arrow[urr, bend left] \arrow[drr, bend right] & X \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & Y
\end{tikzcd}
$$

i.e given $f_i : Y \to X_i$ for all $i$, there exists unique $f : Y \to X$ such that $\pi_i \circ f = f_i$.
In this case we write $f = f_1 \times f_2$.
Note that the product is unique upto isomorphism by initial object argument.
Often we abuse notation and just call $X$ the product.

Products exsit in Set, Top, Ring, Group (take $X = X_1 \times X_2$ the cartesian product) but not in Field.

# Monday, 3/3/2025

Functions into a product are easy since they're determined by components.
Example: curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (\gamma_1(t), \gamma_2(t))$.

**Definition** (Infinite Product)**.** Suppose we have $\{X_j\}_{j \in J}$, ie we have $J \to \mathrm{Ob}\,\mathcal{C}$, we call $J$ an index set. Then, the product of $\{X_j\}_{j \in J}$ is defined similarly:

$$\Big(X, \{\pi_j : X \to X_j\}_{j \in J}\Big)$$

Such that,

$$\begin{tikzcd}
 & & X_j \\
Y \arrow[rru, bend left] \arrow[r, dotted, "\exists!"] & X \arrow[ru] &
\end{tikzcd}$$

When $\mathcal{C} = \text{Top}$ we use the product topology.
Reversing all arrows give us coproducts also known as direct sum:

**Definition.** Coproduct of two objects $X_1, X_2$ is given by a triple:
$\left(X, X_1 \xrightarrow{i_1} X, X_2 \xrightarrow{i_2}\right)$ such that $X$ is final.

$$\begin{tikzcd}
 & & X_1 \arrow[lld, bend right] \arrow[ld, "i_1"] \\
Y & X \arrow[l, dotted, "\exists!"'] & \\
 & & X_2 \arrow[llu, bend left] \arrow[lu, "i_2"']
\end{tikzcd}$$

We write $f = f_1 + f_2$.
Coproducts exist in Set, Top, Ab, Grp, Ring, CRing.
Coproducts are unique upto isomorphism.0

In Ab, the coproduct is given by the direct product, $X = X_1 \times X_2, X_1 \to X, X_2 \to X$ by $a \to (a, 0)$ and $b \to (0, b)$.
$A_1 \oplus A_2 = A_1 \times A_2$.
In Set and Top coproducts are called disjoint union.
$X_1 \coprod X_2, i_1, i_2$
In Set, if $X_1, X_2$ are disjoint then let $X_1 \coprod X_2 = X_1 \cup X_2$.
In general, let $X_1 \coprod X_2 = X_1 \times \{1\} \cup X_2 \times \{2\} \subset X_1 \cup x_2 \times \{1, 2\}$
In Top, if $\overline{X_1}, \overline{X_2}$ are disjoint then we can deinfe $X_1 \coprod X_2 = X_1 \cup X_2$.
In general, consider set theoretic disjoint union $X_1 \coprod X_2, i_1, i_2$ and define $U \subset X_1 \coprod X_2$ open if and only if by definition $i_1^{-1}U, i_2^{-1}U$ open.
Coproducts allow us to have an abstract definition of a Delta commplex.

$$X = \frac{\coprod(J_n \times \Delta^{n-1})}{\sim}$$

In Grp, coproduct is the so calle free product.
In CRing, $X_1 \otimes_{\mathbb{Z}} X_2$
In Ring, free product.
Now we talk about ‘based spaces’ in $\text{Top}_*$.
Objects $(X, x_0)\, x_0 \in X$.
Morphisms $(X, x_0) \to (Y, y_0)$ with $X \xrightarrow{cont} Y, x_0 \mapsto y_0$.
Fund group functor $\pi_1\ \text{Top}_* \to \text{Grp}$. We often write $X$ for $(X, x_0)$.
Coproduct in $\text{Top}_*$ is the wedge sum (one point union) given by $X \vee Y = \frac{X \coprod Y}{x_0 \sim y_0}$.
Example: Figure 8 is $S^1 \vee S^1$.

**Definition.** To apply excision, we want $(X, x_0)$ to be well pointed.
If $x_0$ is closed and $\subset$ open $U$ and $x_0$ is a deformation retract of $U$ then $(X, x_0)$ is well pointed.

**Lemma 49.** a) $H_i X \oplus H_i Y \xrightarrow[\approx]{H_i(i_1 \coprod i_2)} H_i(X \coprod Y)$.

b) If $(X, x_0)$ and $(Y, y_0)$ are well pointed then,

$$\widetilde{H}_i(X) \oplus \widetilde{H}_i(Y) \xrightarrow{\approx} \widetilde{H}_i(X \vee Y)$$

c) If $(X, x_0)$ and $(Y, y_0)$ are welll pointed then $\pi_i X * \pi_i Y \xrightarrow{\approx} \pi_i X \vee Y$

## Wednesday, 3/5/2025

Today we do Pushouts and Adjunction Spaces.
Suppose we have a category $\mathcal{C}$.

**Definition.** A Span is two morphisms with the same domain.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 \end{tikzcd}$$

A Cospan is two morphisms with the same Codomain.

$$\begin{tikzcd} X_1 \arrow[dr] & & X_2 \arrow[dl] \\ & P & \end{tikzcd}$$

**Definition** (Pushout)**.** A pushout of Span:

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 \end{tikzcd}$$

Is a Cospan:

$$\begin{tikzcd} X_1 \arrow[dr] & & X_2 \arrow[dl] \\ & P & \end{tikzcd}$$

which is 'initial': we have the following commutative square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & P \end{tikzcd}$$

Initial in the sense that:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_1 \arrow[d] \arrow[ddr, bend left] & \\ X_1 \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dotted, "\exists!"] & \\ & & Y \end{tikzcd}$$

Sometimes we write a pushout square as follows:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_1 \arrow[d] \\ X_2 \arrow[r] & P \end{tikzcd}$$

**Definition.** We obtain the pullback by reverting the arrows:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X_1 \arrow[d] \\ X_1 \arrow[r] & X \end{tikzcd}$$

Pushouts existt in Set, Top, Ab, Grp, Ring, CRing. It is 'quotient of coproducts' eg in Set and Top,

$$\begin{tikzcd} X \arrow[r, "g_1"] \arrow[d, "g_2"] & X_1 \arrow[d] \\ X_2 \arrow[r] & P \end{tikzcd}$$

$$P = \frac{X_1 \coprod X_2}{g_1(x) \sim g_2(x) \forall x \in X}$$

We write $P = X_1 \cup_X X_2$.
In Ab:
$P = \frac{X_1 \oplus X_2}{\langle g_1(x) - g_2(x) | x \in X \rangle}$
Grp: $X_1 *_X X_2$
TOP:

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_1 \arrow[d] \\ X_2 \arrow[r] & P = X_1 \coprod X_2 \end{tikzcd}$$

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & P = X_1 \vee X_2 \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ X_1 \arrow[r] & P = X_1 / X \end{tikzcd}$$

Now, suppose $U, V$ are open subsets of $X$ or $U, V$ are closed subsets of $X$.

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & P = U \cup V \end{tikzcd}$$

The universal property follows from 'pasting lemma'.

**Theorem 50** (Seifert-van Kampen Theorem)**.** If $X = U \cup V$, both open and $x_0 \in U \cap V$, $U \cap V, U, V$ all path-connected then, the following is a pushout square

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \pi_1(U, x_0) \arrow[d] \\ \pi_1(V, x_0) \arrow[r] & \pi_1(X, x_0) \end{tikzcd} = \pi_1 U *_{\pi_1 U \cap V} \pi_1 V$$

Slogan: $\pi_1$ preserves pushouts.

**Definition.** $f : X \to Y$ is an <u>embedding</u> if $f : X \to f(X)$ is a homeomorphism. We write $X \hookrightarrow Y$.
If $X \subset Y$ we have an induced embedding.

**Definition** (Adjunction Spaces)**.** (See DK). $h : A \to B$ where $A \subset C$, we can look at:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, hook] & B \arrow[d] \\ C \arrow[r] & B \cup_h C \end{tikzcd} = \frac{B \coprod C}{h(a) \sim a \forall a \in A}$$

$B \cup_h C$ is called the adjunction space.
Slogan: Attach $C$ to $B$ along $A$.

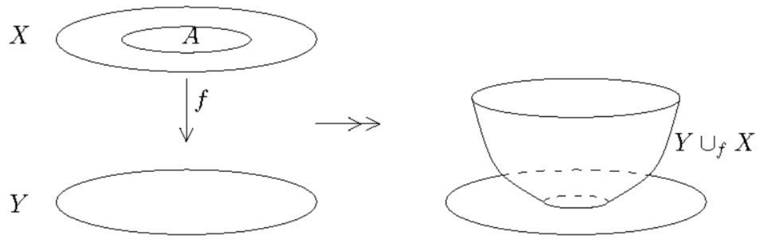



Figure 2: Adjunction Space

Exercise from DK:
If $A$ is closed in $C$,
$B \hookrightarrow B \cup_h C$
$C - A \hookrightarrow B \cup_h C$
Underlying set of $B \cup_h C$ is $B \coprod (C - A)$.
Consider the following examples:

$$\begin{tikzcd} * \arrow[r] \arrow[d, hook] & B \arrow[d] \\ C \arrow[r] & B \vee C \end{tikzcd} \quad = \quad B \cup_{h:*\to B} C$$

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & * \arrow[d] \\ C \arrow[r] & C/A \end{tikzcd} \quad = \quad * \cup_{h:A\to *} C$$

**Definition.** $n$-cell is space homeomorphic to $\mathring{D}^n$

**Definition.** $X \cup_{\phi:S^{n-1}\to X} D^n$ attaches $n$-cell to $X$.

Write $X \cup e^n = X \cup_\phi D^n$
$e^n$ open disk $\subset X \cup e^n$
$B = I^2$ square, $n = 2, \phi : S^1 \to I^2$.

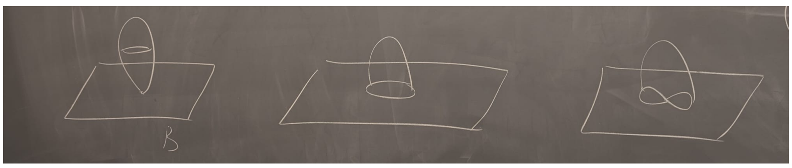



Figure 3: Gluing n-cell on square

**Definition** (CW Complex)**.** (Not complete definition) Built from $\varnothing$ by attaching 0-cells, 1-cells, 2-cells etc.

# Monday, 3/10/2025

Recall: pushout:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\ C \arrow[r] & P \end{tikzcd}$$

$P$ is supposed to be initial. In the Top categry we can have the following construction:

$$P = \frac{B \coprod C}{f(a) \sim g(a) \forall a \in A}$$

**Definition** (Adjunction Space)**.**
$$\begin{tikzcd}
S^{n-1} \arrow[r, "\phi"] \arrow[d, hook] & X & \\
D^n \arrow[r, "\chi"] & X \cup_\phi D^n = & X \cap e^n
\end{tikzcd}$$

**Definition** (CW Complex)**.** Definition A: Constructive, Start with 0-cells, 1-cells, 2-cells etc.
CW complex is a pair $(X, \{X^n\}_{n=0,1,2,\cdots})$ where $X$ is a topological space and $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X$ is an increasing sequence of subspaces of $X$ with the following properties:

1) $X^0$ discrete

2) $\exists$ a pushout diagram of the following form:

$$\begin{tikzcd}
\coprod S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X^{n-1} \arrow[d] \\
\coprod D^n \arrow[r] & X^n
\end{tikzcd}$$

3) $X = \bigcup_{n \geq 1} X^n$

4) 'Weak topology' $A \subset X$ open iff $A \cap X^n \subset X^n$ open $\forall n$.

In this definition, $X^n$ is called the $n$-skeleton of $X$.

**Remark.** Axiom 4 is unnecessary if $\dim X$ is finite, which means we stop adding on sets after a while thus $X = X^n$ for some $n$. $\phi$ are called attaching map, $\chi$ characteristic map.

Examples: $\dim X = 1 \iff X$ is a graph.
If $X$ is a torus, then $X^0$ is a point, $X^1$ is the figure $\infty$ and $X^2$ is the whole torus.

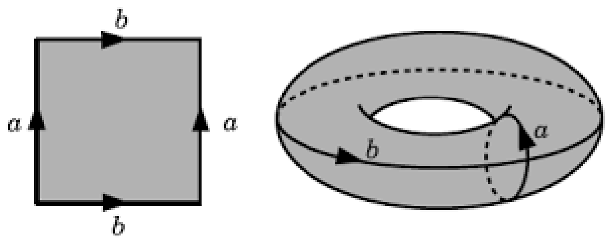



Figure 4: Torus CW Complex

Here $\phi = b^{-1}a^{-1}ba$. Then,

$$\begin{tikzcd}
S^1 \arrow[r, "\phi"] \arrow[d, hook] & \infty \arrow[d] \\
D^2 \arrow[r] & \text{torus}
\end{tikzcd}$$

There's a more elegant way of expressing 4, which is:
$X - \operatorname{colim}_{n \to \infty} X^n$.

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[drr] \arrow[drrr, bend right] & X^1 \arrow[r] \arrow[dr] & X^2 \arrow[r] \arrow[d] \arrow[dr] & \cdots \\
& & X \arrow[r, dotted, "\exists!"] & Y
\end{tikzcd}$$

$\implies$ function $X \to Y$ such that $X^n \to Y$ is continuous then $X \to Y$ is continuous.
$S^n$ has two natural CW structures.
$X^0 = pt = X^1 = X^2 = \cdots = X^{n-1} \subset X^n = S^n = X^{n+1}$.
$S^n = e^0 \cup e^n$. We can also write:
$S^n = e^0_+ \cup e^0_- \cup e^1_+ \cup e^1_- \cup \cdots \cup e^n_+ \cup e^n_-$.
$D^n = e^0 + e^{n-1} \cup e^n$

$$\begin{tikzcd}
S^{n-1} \arrow[r, "ID=\phi"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S^{n-1} \arrow[d] \\
D^n \arrow[r] & D^n
\end{tikzcd}$$

$S^\infty = \cup S^n = \{(x_0, \cdots) \in \mathbb{R}^\infty\}, \left|\sum_i x_i^2\right| = 1$.
$\vee S^1 = \frac{\coprod S^1}{\sim}$

**Definition.** Definition B: Union of disjoint cells
(NOT DISJOINT UNION OF CELLS!!!).
CW complex is a pair $(X, \{e^n_\alpha\})$ is a Hausdorff space $X$ with cells $e^n_\alpha \subset X$ such that:

1) $\forall x \in X, \exists! e^n_\alpha \ni x$ [i.e. $X$ is union of disjoint cells]
2) $\forall e^n_\alpha, \exists \chi^n_\alpha : D^n \to \overline{e^n_\alpha}$ such that $\chi^n_\alpha\big| : \mathring{D}^n \xrightarrow{\approx} e^n_\alpha$ homeomorphism.
3) Closure finite. $\overline{e}^n_\alpha \to e^n_\alpha$ is contained in a finite union of cells of dim $< n$.
4) Weak topology: $A \subset X$ closed $\iff$ $A \cap \overline{e}^n_\alpha \subset \overline{e}^n_\alpha$ [closed food sells]

This is by JHC Whiteheead.
Def A $\implies$ Def B: $X^n - X^{n-1}$ is disjoint set of $n$-cells.
Def B $\implies$ Def A: $X^n = \cup$ cells of dim $\leq n$.

# Wednesday, 3/12/2025

**Proposition 51.** A finite CW complex is compact

*Proof.* $X = \cup \chi(D^n_\alpha)$ then $X$ is finite union of compact spaces $\implies$ $X$ is compact. $\square$

**Proposition 52.** Converse: A compact CW complex is finite.

Idea: Let $K \subset X$ consist of a point in each cell then $K$ is discrete $\implies$ $K$ is closed $\implies$ $K$ is compact $\implies$ $K$ is finite.
$\dot{e}^n = \overline{e^n} - e^n$ boundary.
$\dot{e}^n \subset$ finite union of cells of dim $\leq n - 1$.
Example of CW comple is delta complex.
$\left(X, \left\{\sigma_\alpha : \mathring{\Delta}^n \to X \text{ cells}\right\}\right)$

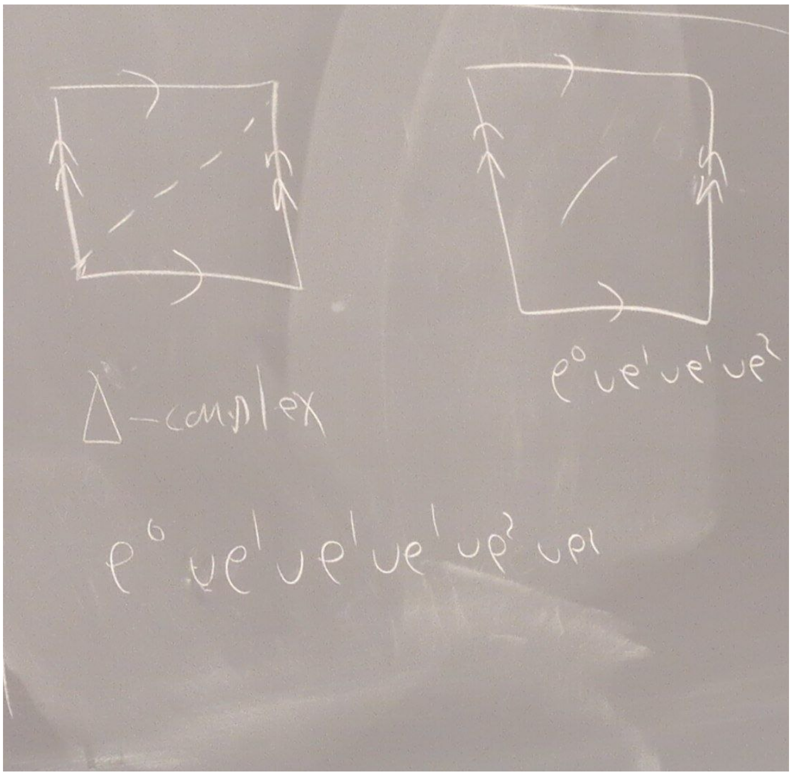



Figure 5: CW Complex, Delta Complex

$$\begin{tikzcd}
S^1 \arrow[r, "\text{awful cont. map}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S^1 \arrow[d] \\
D^2 \arrow[r] & X
\end{tikzcd}$$

$X$ CW complex, not $\Delta$-complex.

**Definition.** A subcomplex of a CW complex is a subset $A \subset X$ such that $(A, \{A \cap X^n\})$ CW complex or $\left(A, \{e^n_\alpha\}_{e^n_\alpha \subset A}\right)$ CW complex.

**Proposition 53.** Suppose $K \subset X, (X, \{X^n\})$ a CW complex.
Then, $K$ is compact $\iff$ $K$ closed, $K \subset A \subset X$ finite subcomplex.

# Cellular Homology

Suppose we have a CW complex $(X, \{X^n\})$.
Cellular chain complex $C_\bullet X = \cdots \to C_{n+1}X \to C_n X \to C_{n-1}X \to \cdots$.
Here, cellular chain $C_n X \coloneqq H_n(X^n, X^{n-1}) \underset{!}{\cong} \mathbb{Z}^{\#\text{ of }n\text{-cells}}$. $C_n X$ is called the relative singular homology.
We also need boundary map. $C.T.^2 = \mathbb{Z}e^2 \xrightarrow{0} \mathbb{Z}e^1_a + \mathbb{Z}e^1_b \to \mathbb{Z}e^0$
We have the following LES:
$C_n X = H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}X^{n-1} \xrightarrow{j} H_{n-1}(X^{n-1}, X^{n-2})$
$\partial^{cw}(z) = [\partial z]$.
Claim: $C_\bullet X = (C_* X, \partial^{cw})$ is a chain complex.

$$\begin{tikzcd}
& & & H_{n-1}X^{n-1} \arrow[dr] & \\
H_{n+1}(X^{n+1}, X^n) \arrow[rr] \arrow[dr, "\partial"'] & & H_n(X^n, X^{n-1}) \arrow[rr] \arrow[ur] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& H_n X^n \arrow[ur, "j"] \arrow[uurr, bend right, "=0"'] & & &
\end{tikzcd}$$

Claim: $(X^n, X^{n-1})$ is a good pair.
Proof: $X^{n-1} \subset X^{n-1} \cup_{\coprod \chi} \coprod D^n - \{0\} \subset X^n$
$C_n X = H_n(X^n, X^{n-1}) \cong \widetilde{H}_n(X^n/X^{n-1}) \cong \widetilde{H}_n(\vee S^n) \cong \oplus_{n\text{-cells}} \mathbb{Z}$

## Friday, 3/14/2025

We talk about why $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$
Recall: good pair $\implies H_*(D^n, S^{n-1}) \cong \widetilde{H}_*(D^n/S^{n-1}) \cong \widetilde{H}_*(S^n) \cong (0, \cdots, \mathbb{Z}, 0, \cdots)$
What are the generators?
$\mathrm{id} : \Delta^n \to \Delta^n \in H_n(\Delta^n, \partial\Delta^n) = H_n(D^n, S^n)$
We have the notion of the boundary of a cell, $\dot{e}^n = \overline{e^n} - e^n$

**Theorem 54** (Isomorphism Theorem)**.** The following maps are isomorphisms.

i) $H_n(D^n, S^{n-1}) \xrightarrow{\chi_*} H_n(\overline{e^n}, \dot{e}^n) \xrightarrow{q_*} \widetilde{H}_n(\overline{e^n}/\dot{e}^n) \to \widetilde{H}_n(D^n/S^{n-1}) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z}$

ii) $\bigoplus_{I_n} H_n(D^n, S^{n-1}) \to H_n\left(\coprod D^n, \coprod S^{n-1}\right) \xrightarrow{\chi_* + q} H_n(X^n, X^{n-1}) \xrightarrow{q_*} \widetilde{H}_n(X^n/X^{n-1}) = \widetilde{H}_n(\vee S^n) \cong \bigoplus_{I_n} \widetilde{H}_n(S^n) \cong \bigoplus_{I_n} \mathbb{Z}$

iii) $\bigoplus_{\alpha \in I_n} H_n(\overline{e}^n_\alpha, \dot{e}^n_\alpha) \xrightarrow{\cong} H_n(X^n, X^{n-1}) = C_n X$

*Proof.* i and ii: $q_*$ arre isomorphism because good pairs.
Similarly, $q_* \circ \chi_*$ is an isomorphism.
iii follows from i and ii. $\square$

We also have basis for $C_n X$, it is the image of $H_n(\overline{e}^n_\alpha, \dot{e}^n_\alpha)$s.

**Definition.** Oreintation for $e^n_\alpha$ is a choice of generator $[e^n_\alpha] \in H_n(\overline{e}^n_\alpha, \dot{e}^n_\alpha)$.
$\{[e^n_\alpha]\}_{\alpha \in I_n}$ give a basis for $C_n X$
'oriented $n$-cells'
It's ok (???) to be sloppy and write $e^n_\alpha \in C_n X$.

We have basis for $C_n X$ and $C_{n-1} X$ given by $[e^n_\alpha]$ and $[e^{n-1}_\beta]$.
$\partial[e^n_\alpha] = \sum_\beta \delta_{\beta\alpha}[e^{n-1}_\beta]$
We want to find the matrix of $\partial$ where $C_n X \xrightarrow{\partial} C_{n-1} X$. We essentially want to find $\partial_{\beta\alpha}$.
'Degree of attaching maps'

$$S^{n-1}_\alpha \xrightarrow{\phi_\alpha} X^{n-1} \to X^{n-1}/X^{n-2} = \vee_{I_{n-1}} S^{n-1} \to S^{n-1}_\beta$$

This is a map of sphere! Then we have degree, so we set $\deg = \partial_{\beta\alpha}$.
Example: Torus $T^2$ has CW structure $e^0 \cup e^1_a \cup e^1_b \cup e^2$.
We choose orientation on $e^2$.
$C_2 T^2 = \mathbb{Z}[e^2] \xrightarrow{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \mathbb{Z}[e^1_a] \oplus \mathbb{Z}[e^1_b] \to \mathbb{Z}[e^0]$.
$\partial[e^2] = [e^1_a] + [e^1_b] - [e^1_a] - [e^1_b] = 0$.

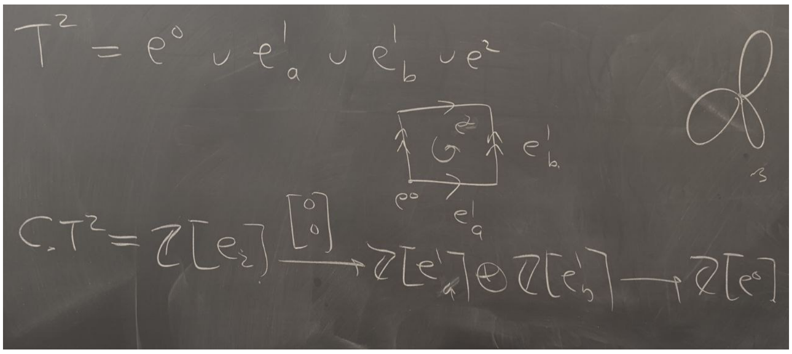



Figure 6:

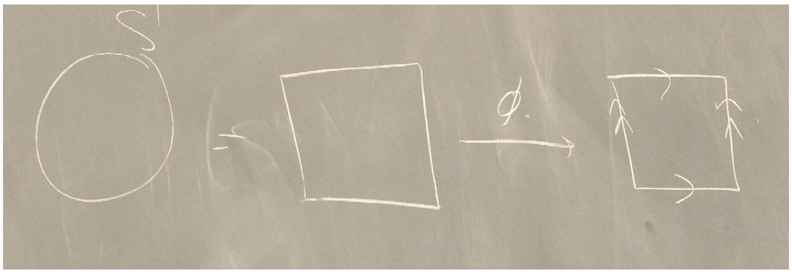



Figure 7:

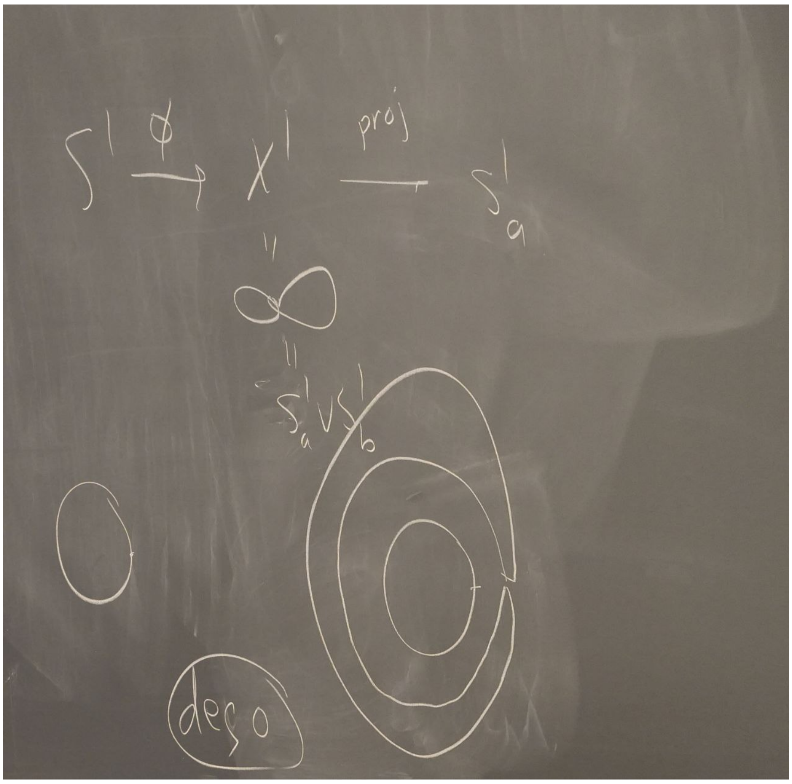



Figure 8:

So, $S^1 \xrightarrow{\phi} X^1 \xrightarrow{proj} S^1_a$, since $X^1 = S^1_a \cup S^1_b$, the projection goes around $S^1_a$, then waits at $S^1_b$ since that's collapsed to a point, and then it goes back. This map has $\deg 0$.

## Projective Space

$\mathbb{R}P^n$ = space of lines through 0 in $\mathbb{R}^{n+1}$
$\mathbb{C}P^n$ = space of $\mathbb{C}$-lines through 0 in $\mathbb{C}^{n+1}$
This is difficult so we take quotient space.
$\mathbb{R}P^n \cong \frac{\mathbb{R}^{n+1}-\{0\}}{\underline{x}\sim\lambda\underline{x}} \cong \frac{S^n}{\underline{x}\sim-\underline{x}} \cong \frac{D^n}{\underline{x}\sim-\underline{x} \text{ when } x \text{ is on the boundary}}$.
Then the simplest way to look at $\mathbb{R}P^2$ is the disk $D^2$ where we identify opposite points.
Same logic gives us $\mathbb{R}P^1 \cong S^1$.
So, we have $S^n \to \mathbb{R}P^n$ double cover.

## Monday, 3/24/2025

Was absent

## Wednesday, 3/26/2025

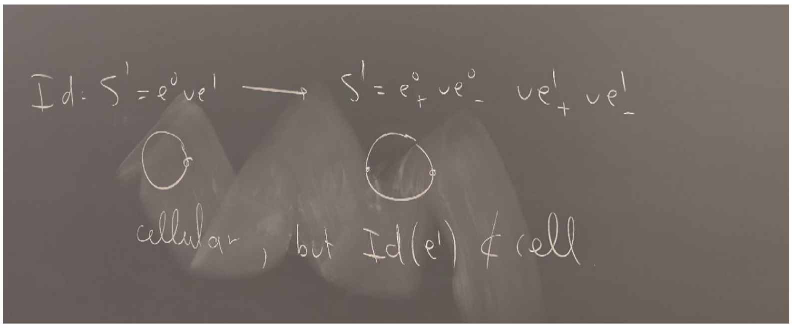
## Cellular Maps

$H_*^{\mathrm{CW}} \cong H_*^{\mathrm{sing}} X$
Morphisms are people too:
Suppose we have CW $(X, \{X^n\}), (Y, \{Y^n\})$

**Definition.** Continuous $f : X \to Y$ is <u>cellular</u> if:

$$f(X^n) \subset Y^n$$

Example: $\mathrm{id} : S^1 \to S^1$ but with two different CW decomposition:




Thus, $\not\Longleftrightarrow$ $f(n\text{-cell}) \subset n\text{-cell}$.
Then chain map $f_\# = C_\bullet f : C_\bullet X \to C_\bullet Y$

$$C_n X = H_n(X^n, X^{n-1}) \to H_n(Y^n, Y^{n-1}) = C_n Y$$

$\implies f_* = H_* f : H_*^{\mathrm{CW}} \to H_*^{\mathrm{CW}} Y$
Thus we have the Category CW
Object: CW complexes.
Morphism: Cellular map
We have functors:

$$\begin{tikzcd} CW \arrow[r, "C_\bullet"] \arrow[rr, bend right, "H_*^{\mathrm{CW}}"'] & Ch \arrow[r, "H_*"] & Gr_{\mathbb{Z}} \end{tikzcd}$$

**Theorem 55** (Cellular Approximation Theorem, CAT)**.** Every continuous map $f : X \to Y$ is homotopic to a cellular map.

There is a relative version of it:
If $A$ is a subcomplex of $X$ and $f\big|_A$ is cellular then $f \simeq$ cellular (rel $A$). Relative to $A$ meaning the homotopy map is identity on $A$, meaning $H(a,t) = H(a,0)$.
Proof ommitted.
Examples:


This is not cellular. But we can squeeze the bottom part into the point. So homotopic to a cellular map.
id $\simeq f$ cellular.
$f((S^1)^0) \subset (S^1)^0$.
$f((S^1)^1) \subset (S^1)^1$
Suppose $k < l$. Every $g : S^k = e^0 \cup e^k \to s^l e^0 \cup e^l$ is homotopic to a cellular map, i.e. constant map.
i.e. $\pi_k S^l = 0, \pi^1 S^2 = 0$.

**Theorem 56.** Suppose we have CW $(X, \{X^n\})$.
$H_*^{\mathrm{CW}} X \cong H_* X$.

We need three lemmas.
L1: $H_i(X^n, X^{n-1}) \equiv 0$ for $i \neq n$.

*Proof.* Slogan: Good pair. $H_*(X^n, X^{n-1}) \cong \widetilde{H}_*(X^n/X^{n-1}) = \widetilde{H}_*(\vee S^n)$. $\square$

Corollary: $H_i X^{n-1} \cong H_i X^n$ when $i \neq n-1, n$.

*Proof.* LES of pairs $\square$

L2: $H_n X^{n-1} = 0$

*Proof.* $H_n X^0 \xrightarrow{\approx} \cdots \xrightarrow{\approx} H_n X^{n-2} \xrightarrow{\approx} H_n X^{n-1}$ $\square$

L3: $H_n X^{n+1} \xrightarrow{\approx} H_n X$

*Proof.* Compactness. If we have $\sigma : \Delta^k \to X$ then $\sigma(\Delta^k) \subset X^N$ for some $N$, compact.
Any compact set intersects only a finite number of cells.
$\implies S_\bullet X = \bigcup_N S_\bullet X^N$.
We check $H_n X^{n+1} \to H_n X$ onto.
Take $[\alpha] \in H_n X \implies \exists N$ s.t. $\alpha \in S_n X^N$
Thus, $H_n(X^{n+1}) \simeq H_n(X^N) \to H_n X$ where the map is given by $[\alpha] \mapsto [\alpha]$.
Injectivity: suppose $[\alpha] \mapsto 0$ then $\alpha \in \partial\beta \implies \beta \in S_{n+1} X^N$ for some $N$.
$H_n X^{n+1} \to H_n X^N$ is an isomorphism, but here $[\alpha] \mapsto 0$. Contradiction. $\square$

We prove the original theorem by combining L1, L2, L3.

*Proof.* Diagram Chase!

$$
\begin{tikzcd}
 & & & & & H_n(X^{n+1},X^n) \overset{L1}{=} 0 \\
 & 0 \underset{L2}{=} H_n X^{n-1} \arrow[dr] & & H_n(X^{n+1}) \overset{L3}{=} H_n X \arrow[urr] & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j"] & & & \\
 & \underset{=C_{n+1}X}{H_{n+1}(X^{n+1},X^n)} \arrow[ur, "\partial"] \arrow[rr, "\partial^{\mathrm{CW}}_{n+1}"] & & \underset{=C_nX}{H_n(X^n,X^{n-1})} \arrow[rr, "\partial n^{\mathrm{CW}}"] \arrow[dr] & & \underset{=C_{n-1}X}{H_{n-1}(X^{n-1},X^{n-2})} \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, tail] & \\
 & & & 0 \underset{L2}{=} H_{n-1}X^{n-2} \arrow[ur] & &
\end{tikzcd}
$$

We have commutative, exact diagonals. Thus,
$j : H_nX^n \rightarrowtail \ker \partial_n^{\mathrm{CW}}$
$j : \operatorname{im} \delta \rightarrowtail \operatorname{im} \partial_{n+1}^{\mathrm{CW}}$
Then,

$$H_n^{\mathrm{CW}} X = \frac{\ker \partial^{\mathrm{CW}}}{\operatorname{im} \partial^{\mathrm{CW}}} = \frac{H_nX^n}{\operatorname{im} \partial} = \operatorname{cok} \partial \cong H_n X$$

□

# Friday, 3/28/2025

# Euler Characteristic

$\chi(S^{2k}) = 2, \chi(S^{2k+1}) = 0$
$\chi(\mathbb{R}P^2) = 1, \chi(T^2) = 0$
If we have an abelian group $A$:

**Definition.** $\{v_1, \cdots, v_k\} \subset A$ are <u>linearly independent</u> if:

$$n_1v_1 + \cdots n_kv_k = 0 \implies n_1 = n_2 = \cdots = n_k = 0$$

**Definition.** $\operatorname{rank} A = \sup\{k \mid \exists \text{lin. ind.} \{v_1, \cdots, v_k\}\}$
$= \sup\{k \mid \exists \mathbb{Z}^k \hookrightarrow A\}$

Example: $\operatorname{rank} \mathbb{Z} = 1$
$\operatorname{rank} \mathbb{Z}^k = k$
$\operatorname{rank} \mathbb{Z}^k \oplus \text{torsion} = k$
$\operatorname{rank} \mathbb{Q} = 1$
$\operatorname{rank} \mathbb{R} = \infty$
$\operatorname{rank} \mathbb{Q}/\mathbb{Z} = 0$
$\operatorname{rank} \mathbb{R}/\mathbb{Z} = \infty$
If we have subgroup s.t. $[A : B] < \infty$ then $\operatorname{rank} A = \operatorname{rank} B$.

**Definition** (Euler Characteristic)**.** For a space $X$,

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i X$$

Defined if $\operatorname{rank} H_iX < \infty \forall i, \operatorname{rank} H_iX = 0$ for $i \gg 0$.

$\chi(S^2) = 1 - 0 + 1 = 2$
$\chi(\text{figure 8}) = 1 - 2 = -1$

**Definition** (Betti Numbers)**.**

$$\beta_i(X) \coloneqq \operatorname{rank} H_i X$$

**Lemma 57** (Additivity Lemma)**.** If we have a SES:

$$0 \to A \to B \to C \to 0$$

Then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

*Proof.* Special Case: suppose $C$ is finitely generated. Then $C = \mathbb{Z}^k \oplus \text{finite}$. Let $B \xrightarrow{\pi} C$. From the SES, we can restrict so that:

$$0 \to A \to \pi^{-1}\mathbb{Z}^k \to \mathbb{Z}^k \to 0$$

Thus, $\pi^{-1}\mathbb{Z}^k \cong A \oplus \mathbb{Z}^k$.
Also: finite index $\implies \operatorname{rank} B = \operatorname{rank} \pi^{-1}\mathbb{Z}^k = \operatorname{rank} A + \operatorname{rank} \mathbb{Z}^k = \operatorname{rank} A + \operatorname{rank} C$.
General case: Use two things: $\operatorname{rank} A = \dim_{\mathbb{Q}} A \otimes \mathbb{Q}$.
$\mathbb{Q}$ is a flat $\mathbb{Z}$-module.
We have the following SES:

$$0 \to A \otimes \mathbb{Q} \to B \otimes \mathbb{Q} \to C \otimes \mathbb{Q} \to 0$$

Thus, $\operatorname{rank} B = \dim B \otimes \mathbb{Q} = \dim A \otimes \mathbb{Q} + \dim C \otimes \mathbb{Q} = \operatorname{rank} A + \operatorname{rank} C$. □

**Theorem 58.** If $C_\bullet = \{0 \to C_n \to \cdots \to C_0 \to 0\}$ is a chain complex [so $\delta \circ \delta = 0$] and $\operatorname{rank} C_i < \infty \forall i$, then,

$$\chi(H_* C) = \chi(C_*)$$

Meaning:

$$\sum_i (-1)^i \operatorname{rank} H_i C = \sum_i (-1)^i \operatorname{rank} C_i$$

*Proof.* Let $Z_i = \text{cycles} = \ker(\partial_i : C_i \to C_{i-1})$
$B_i = \text{boundaries} = n \operatorname{im}(\partial_{i+1} : C_{i+1} \to C_i)$
$H_i = \text{homology} = Z_i / B_i$
We have SES:

$$0 \to B_i \to Z_i \to H_i \to 0$$

$$0 \to Z_i \to C_i \to B_{i-1} \to 0$$

Additivity lemma $\implies$:
$\sum_i (-1)^i \operatorname{rank} H_i = \sum_i (-1)^i (\operatorname{rank} Z_i - \operatorname{rank} B_i) = \sum_i (-1)^i \operatorname{rank} Z_i - \sum_i (-1)^i \operatorname{rank} B_i = \sum_i \operatorname{rank} Z_i + \sum_i (-1)^i \operatorname{rank} B_{i-1} = \sum_i (-1)^i \operatorname{rank} C_i$ □

**Corollary 59.** Corollary 1: Suppose $X$ is a finite CW complex. Then,

$$\sum_i (-1)^i \operatorname{rank} H_i X = \sum_i (-1)^i \# \text{ of } i\text{-cells}$$

eg $\chi(S^2) = v - e + f$

**Corollary 60.** Suppose we have an exact sequence:

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to 0$$

Then, $\chi(C_*) = 0$.

$$\sum_{i \text{ even}} \operatorname{rank} H_i C = \sum_{i \text{ odd}} \operatorname{rank} H_i C$$

**Corollary 61.** If $0 \to C_\bullet \to D_\bullet \to E_\bullet \to 0$ is a chain complex then,

$$\chi(H_* D_\bullet) = \chi(H_* C_\bullet) + \chi(H_* E_\bullet)$$

*Proof.* Zig-Zag lemma and the theorem. $\square$

**Corollary 62.** If $(X, A)$ is a pair, then $\chi(X) = \chi(A) + \chi(H_*(X, A))$

**Corollary 63.** If $(X, A)$ is a good pair, then,

$$\chi(X) = \chi(A) + \chi(X/A) - 1$$

**Theorem 64** (Poincaré-Hopf)**.** Let $M$ be a closed $n$-manifold.

$$\chi(M) = 0 \iff \text{ can comb the hairy manifold}$$

i.e. $\exists$ a nowhere 0 tangent vector field.

# Monday, 3/31/2025

# More Euler Characteristic

Let $\Sigma$ be a surface which is a closed 2-manifold.

i) Orientable surfaces are classified by $\chi$.
   $\chi(S^2) = 2, \chi(T^2) = 0, \chi(T^2 \# T^2) = -2, \chi(T^2 \# T^2 \# T^2) = -4$.
   $\#$ is the connected sum.
   Thus $\chi$ classifies orientable surfaces upto homeomorphism.

ii) Non-orientable surfaces are classified by Euler characcteristic.
   $\chi(\mathbb{R}P^2) = 1, \chi(K = \mathbb{R}P^2 \# \mathbb{R}P^2) = 0, \chi(\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2) = -1, \cdots$.

Surfaces are classified by Euler characteristic and orentiability.
Under connected sum $\#$, the surfaces form a commutative monoid. The generators of this monoid are: $\mathbb{R}P^2, T^2$.
Relation: $\mathbb{R}P^2 \# T^2 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.
Let $\Sigma$ be a Riemannian surface now [meaning we have a metric and can measure lengths].
Classical way: embed $\Sigma^2 \subset \mathbb{R}^3$.
Then we have curvature $K : \Sigma \to \mathbb{R}$.

**Theorem 65** (Gauss-Bonnet Theorem)**.**

$$\int_\Sigma K \, \mathrm{d}S = 2\pi\chi(\Sigma)$$

Let $p \in \Sigma$ be a point. $K > 0$ at $p$ means the surface lies on one side of tangent plane [think a sphere]. $K < 0$ means the surface lies on both sides of a tangent plane [think hyperboloid].
If $K \equiv 1$ that is a sphere. Then $S^2 = 4\pi$.
In a torus, $\int_{T^2} K \, \mathrm{d}S = 2\pi\chi(T^2) = 0$. On the 'outer ring' $K > 0$ and on the inner ring $K < 0$.
Even in weirdo embeddings in $\mathbb{R}^3$ there must be one point with positive curvature.

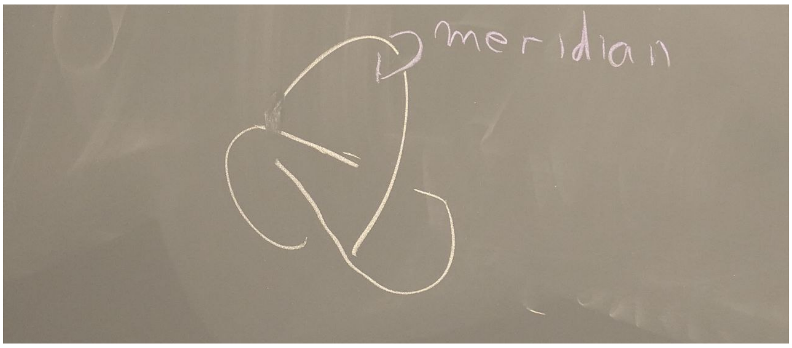
# Hatcher 2B (or not 2 B?)

Goal: Prove Jordan separation theorem.

**Theorem 66.** Let $h : S^{n-1} \hookrightarrow S^n$ [or $\mathbb{R}^n$].
Then $S^n - h(S^{n-1})$ has two components.

Another Goal: Prove Alexander duality.

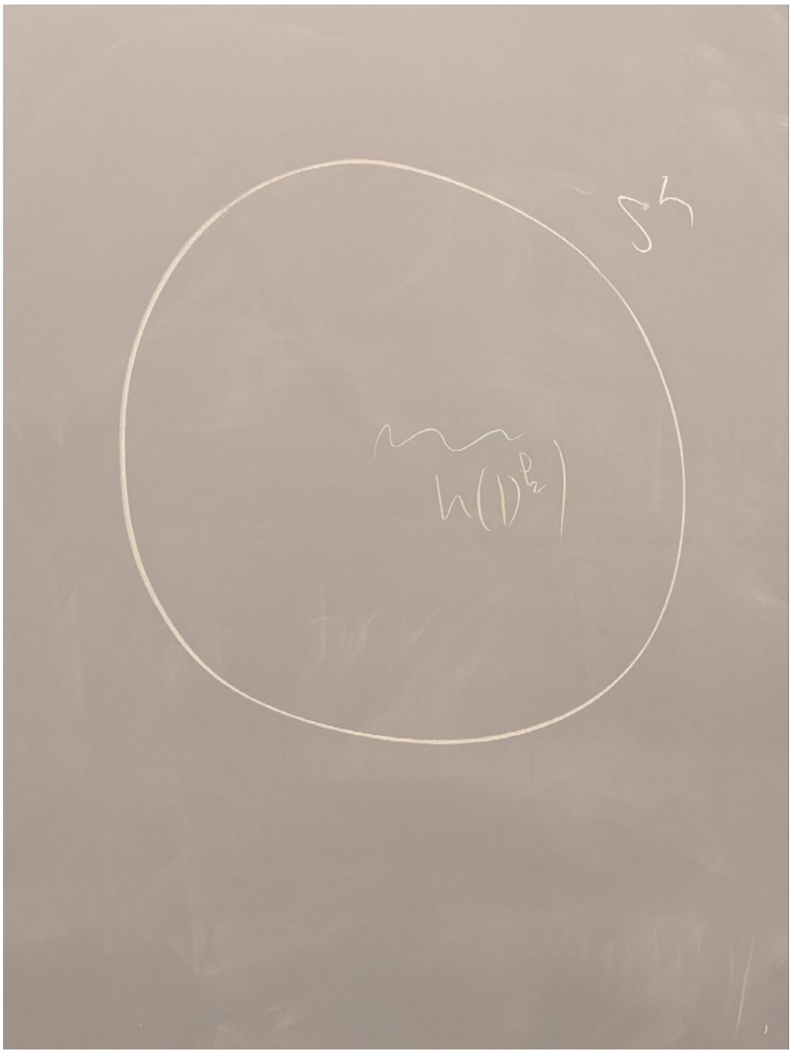
**Theorem 67.** Let $h : S^1 \hookrightarrow S^3$ be a knot. Then $H_1(S^3 - S^1) \cong \mathbb{Z} \cong \mathbb{Z}$.
Also: meridian



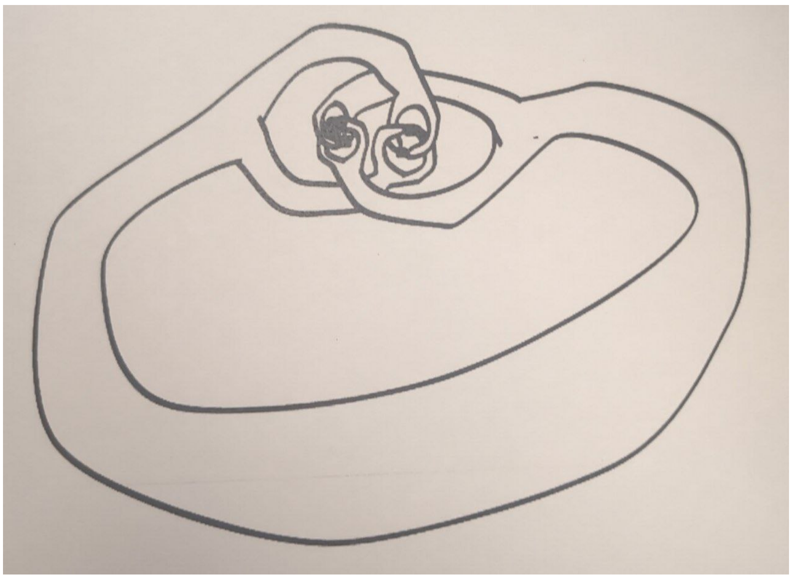

Also: invariance of domain: open $U \subset \mathbb{R}^n$, $h : U \to \mathbb{R}^n$ 1-1, then $U \xrightarrow{\approx} h(U)$.
Also one point compactification: $S^n - \partial \cong \mathbb{R}^n \iff S^n = \mathbb{R}^n \cup \{\infty\}$.

**Theorem 68** (Boring Theorem). $\forall h : D^k \hookrightarrow S^n$ embedding, then,

$$\widetilde{H}_*(S^n - h(D^k)) = 0$$

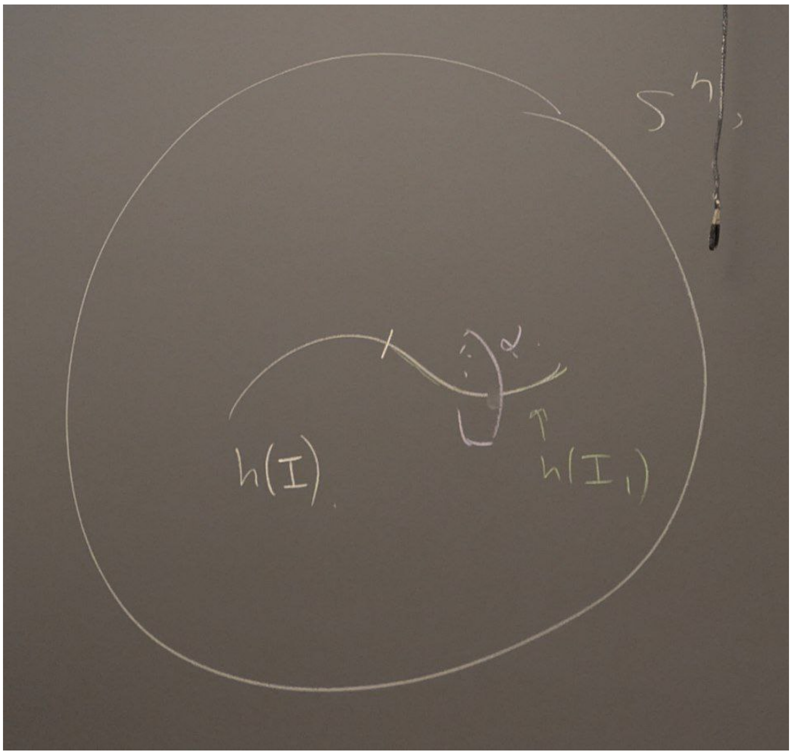


Also: Alexander horned spphere.
$H \cong D^3, \pi_1(\mathbb{R}^3 - H) \neq 1, H_1(\mathbb{R}^3 - H) = 0$. Recursively:
The meridians of horn bound a surface????



*Proof.* (Proof of Boring Theorem) We induct on $k$. $k = 0 : S^n - \text{pt} \cong \mathbb{R}^n$
Assume true for $k - 1$.

Replace $D^k$ by the 'cube' $I^k = [0,1]^k$.
Assume, for the sake of contradiction: there exists a nonzero homology class $0 \neq [\alpha] \in H_i(S^n - h(I^k))$.
Claim 1: we 'chop' into pieces until dimension is $k-1$. Formally: in the $k=1$ case, $\exists$ interval $I = I_0 \supset I_1 \supset I_2 \supset \cdots$ satisfying length of $I_j = \frac{1}{2^j}$ and $0 \neq [\alpha] \in \widetilde{H}_i(S^n - h(I_j))$.




Claim 2L $\forall$ filtration of space $X$ by open sets [ie $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X, X = \cup X_j$],
If $[\alpha] \in \ker(\widetilde{H}_1(X_0) \to H_i X)$ then $\exists j$ such that $[\alpha] \in \ker \widetilde{H}_i(X_0) - \widetilde{H}_i(X_j)$.
Idea: Claim 1 involves MVES, claim 2 involves compactness.
Proof of claim 2: $[\alpha] \in \ker \widetilde{H}_i(X_0) \to \widetilde{H}_i X$
Then $\alpha = \partial\beta, \beta = \sum_k n_k \sigma_k$, $\beta \in S_{i+1}X, \sigma_k : \Delta^{k+1} \to X$.
$\mathrm{supp}(\beta) = \cup \sigma_k(\Delta^{i+1})$ compact.
$X_j \cap \mathrm{supp}(\beta)$ is open cover of $\mathrm{supp}(\beta)$.
Thus $\exists j$ such that $\mathrm{supp}(\beta) \subset X_j \implies [\alpha] \equiv 0 \in \widetilde{H}_i(X_j)$ which is the claim.

## Wednesday, 4/2/2025

We rename some stuff from yesterday. We proved:
Lemma 1: if $X_0 \subset X_1 \subset \cdots \subset X$ is a filtration of $X$ by open sets, and if $[\alpha] \in \ker(\widetilde{H}_i X_0 \to \widetilde{H}_i X)$ then $\exists j$ such that $[\alpha] \in \ker(\widetilde{H}_i X_0 \to \widetilde{H}_i X_j)$.

*Proof.* If $\alpha = \partial\beta, \mathrm{supp}(\beta)$ compact then $\mathrm{supp}(\beta) \subset X_j$ for some $j$. $\square$

Main Proof: Replace $D^k$ by the cube $I^k = [0,1]^k$. We induct on $k$.
True for $k = 0$ since $S^n - \mathrm{pt}$ is homeomorphic to $\mathbb{R}^n$.
Assume true for $k-1$ for induction.
Assume homology is nontrivial for contradiction.
$\exists [\alpha] \in \widetilde{H}_i(S^n - h(I^k))$.
Lemma 2: $\exists$ nested intervals $I_0 = [0,1] \supset I_1 \supset I_2 \supset \cdots$ such that:

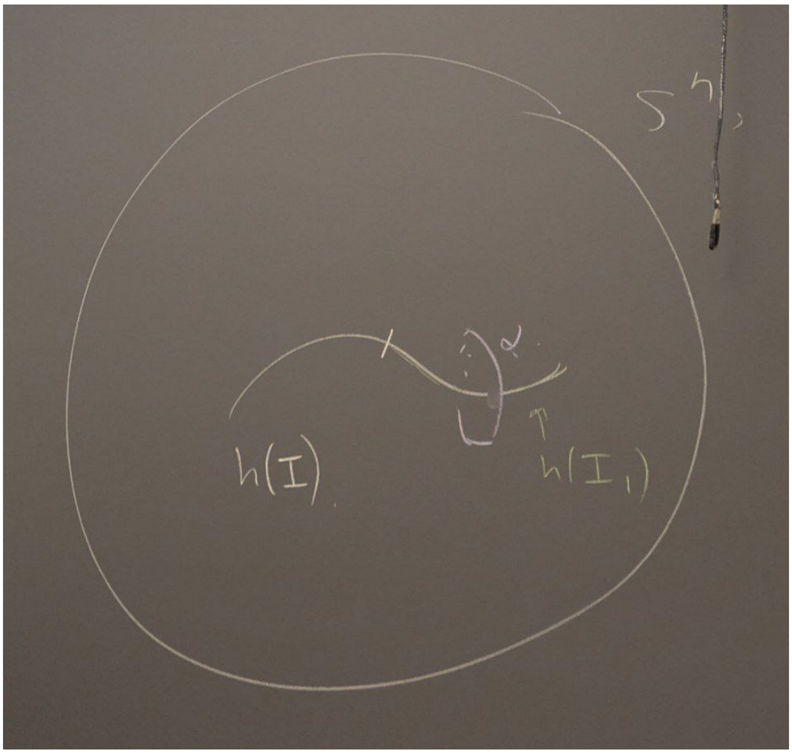
i) length $I_j = \frac{1}{2^j}$

ii) $0 \neq [\alpha] \in \widetilde{H}_i(S^n - h(I^{k-1} \times I_j))$

*Proof.* MVES.
Let $A = S^n - h(I^{k-1} \times [0, \frac{1}{2}])$.
$B = S^n - h(U^{k-1} \times [\frac{1}{2}, 1])$
$A \cup B = S^n - h(I^{k-1} \times \frac{1}{2})$. $\widetilde{H}_*(A \cup B)$ is trivial by induction.
$A \cap B = S^n - h(I^k)$.

$$0 \to \widetilde{H}_i(A \cap B) \xrightarrow{\approx} \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to 0$$

$[\alpha]$ is nontrivial on one so it is nontrivial in at least one of the others.
$0 \neq [\alpha] \in \widetilde{H}_i(A) \implies$ choose $I_1 = [0, \frac{1}{2}]$
$0 \neq [\alpha] \in \widetilde{H}_i(B) \implies$ choose $I_1 = [\frac{1}{2}, 1]$.
Repeat to get $I_2, I_3, \cdots$ □




$\bigcap I_j = \{p\}$
Let $X = S^n - h(I^{k-1} \times p), X_j = S^n - h(I^{k-1} \times I_j)$. $[\alpha] \neq 0 \in \widetilde{H}_i X$ by L2. This contradicts L1.

□

**Theorem 69** (Alexander Duality, Prop 2B1)**.** $\forall h : S^k \hookrightarrow S^n$,

$$\widetilde{H}_*(S^n - h(S^k)) \cong \widetilde{H}_*(S^{n-k-1})$$

'Represented by linking $S^{n-k-1}$'.

Definition of Linking Sphere: Take a disk that intersects the space at one point and take the boundary.

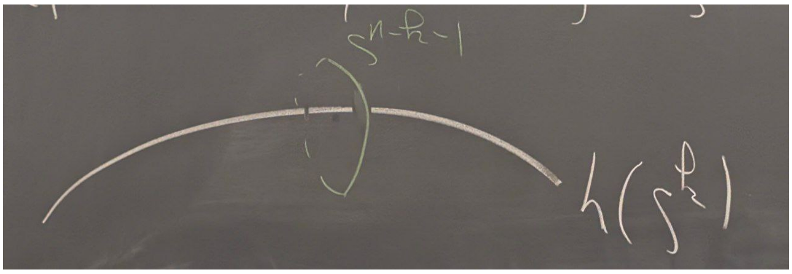



Figure 9: linking sphere

*Proof.* Induct on $k$.
$k = 0, S^n - 2 \text{ pts} \cong S^{n-1} \times \mathbb{R}$, true.

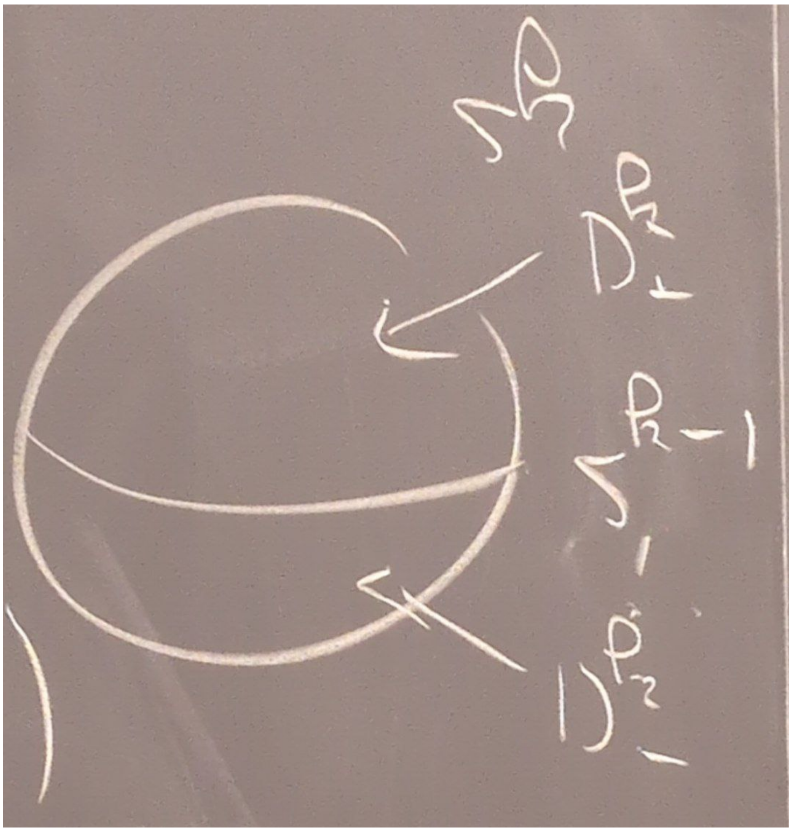



Figure 10:

excisive triad:

$$\begin{array}{ccc} S^n - h(S^k) & \longrightarrow & S^n - h(D^k_+) \\ \downarrow & & \downarrow \\ S^n - h(D^k_-) & \longrightarrow & S^n - h(S^{k-1}) \end{array}$$

$S^n - h(D^k_\pm)$ are acyclic by Boring theorem [homology of a point]
MVES says:

$$0 \oplus 0 \to \widetilde{H}_*(S^n - h(S^{k-1})) \xrightarrow[\partial]{\approx} \widetilde{H}_{*-1}(S^n - h(S^k)) \to 0 \oplus 0$$

□

Special case: If $h : S^{n-1} \hookrightarrow S^n$ then $S^n - h(S^{n-1})$ has two path components, both have homology of point. So $\widetilde{H}_0 = \mathbb{Z}$.

This is Jordan-Brouwer Separation Theorem.

**Remark.** Both components are open. Boundary of each components is $S^{n-1}$.

If $n = 2$ all embeddings are standard: $\exists$ homeo $H : (S^2, S^1) \to (S^2, h(S^1))$.
We have non-standard embeddings:
$D^3 \hookrightarrow S^3, S^2 \hookrightarrow S^3, D^2 \hookrightarrow S^3$.
$S^1 \hookrightarrow S^3$ knots: $H_*(S^3 - \text{knot}) = H_*(S^1)$ but homotopy group might not be same.
We also have nonstandard $D^1 \hookrightarrow S^3$, wild arc.

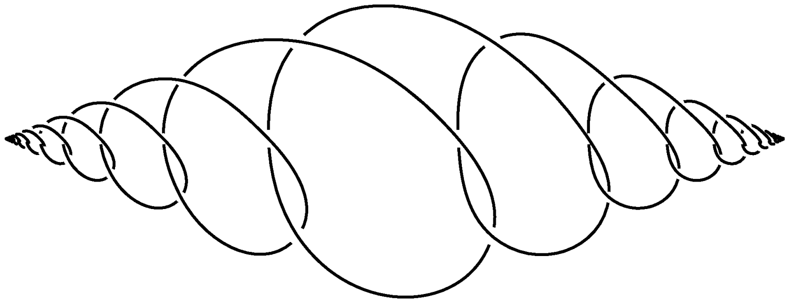


Figure 11: Wild Arc

# Friday, 4/4/2025

Syllabus:
Homology with Coefficients Borsuk-Ulam Ham Sandwich Invariance of Domains Division Algebras Axioms Cohomology

# Homology with Coefficients

Let $M$ be an abelian group ($\mathbb{Z}$-module).
$S_n(X; M)$ singular $n$-chains with coefficients in $M$.
$m_1\sigma_1 + \cdots + m_k\sigma_k \in S_n(X; M), m_i \in M, \sigma_i : \Delta^n \to X$.
i.e. $S_n(X; M) = \bigoplus_{\sigma:\Delta^n \to X} M$ abelian group
Davis-Kirk: $S_n(X; M) = S_n X \otimes_{\mathbb{Z}} M$. $\partial_n^M = \partial_n \otimes \mathrm{id}_M$
Define $\partial_n^M : S_n(X; M) \to S_{n-1}(X; M)$:

$$\partial_n^M(m\sigma) = m\left(\sum_{j=0}^{n}(-1)^j \sigma \circ \delta_n^j\right)$$

As before. The alternating sign implies the double composite is zero, so we have a chain complex $S_\bullet(X; M) = (S_\bullet(X, M), \partial_*^M)$.
$H_n(X; M) := H_n(S_\bullet(X; M)) = \frac{\ker \partial_n^M}{\operatorname{im} \partial_{n+1}^M}$.
Example:
$H_n(X; \mathbb{Z}) = H_n X$.
$H_n(X; M \oplus N) = H_n(X; M) \oplus H_n(X; N)$.
$H_n(X; \mathbb{Z}^k) = (H_n X)^k$.
We actually get the whole package: pair, excision, MV, homotopy invariance, reduced, cellular.
When $X$ is CW, $C_n(X; M) = H_n(X^n, X^{n-1}; M) \cong \widetilde{H}_n(X^n/X^{n-1}, M) \cong M^{\#\text{of } n\text{-cells}}$
So $C_n(X; M) = \bigoplus_j M e_j^i$.
$\partial_n^{CW;M}$ has same matrix as $\partial_n^{CW}$:

$$\partial_n^{CW,M} = \partial_n^{CW} \otimes \mathrm{id}_M$$

Proof: Suppose $X$ is CW. When $d_{\alpha\beta}$ denotes degree:

$$\partial_n^{CW} e_\beta = \sum_\alpha d_{\alpha\beta} e_\alpha$$

Then we have:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\phi_B"] \arrow[rrr, bend right, "d_{\alpha\beta}=\text{degree}"'] & X^n \arrow[r] & X^n/X^{n-1} \arrow[r, "pr_\alpha"] & S^{n-1} \end{tikzcd}$$

In order to prove that $\partial_n^{CW;M}$ has same matrix as $\partial_n^{CW}$ we need lemma 2.49:
Lemma 2.49: If $f : S^k \to S^k$ has degree $d$ then,

$$f_* : H_k(S^k; M) \to H_k(S^k; M)$$

is multiplication by $d$.
To prove this we need the claim: $\partial_n^M(me_B) = m \sum_\alpha d_{\alpha\beta} e_\alpha$.

*Proof.* Note that $H_n : \mathrm{Top} \times \mathrm{Ab} \to \mathrm{Ab}$ given by $(X, M) \mapsto H_n(X; M)$ is a functor.
Let $m \in M$. Let $\Delta_m : \mathbb{Z} \to M$ given by $1 \mapsto m$.
We can consider $(\mathrm{id}, \Delta_m) \circ (f, \mathrm{id}) = (f, \Delta_m) = (f, \mathrm{id}) \circ (\mathrm{id}, \Delta_m)$ which gives a commutative square. So we have:

$$\begin{tikzcd} H_k(S^k; \mathbb{Z}) \arrow[r] \arrow[d] & H_k(S^k; \mathbb{Z}) \arrow[d] \\ H_k(S^k; M) \arrow[r] & H_k(S^k; M) \end{tikzcd}$$

We have the maps:

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & da \arrow[d, mapsto] \\ ma \arrow[r, mapsto] & dma = mda \end{tikzcd}$$

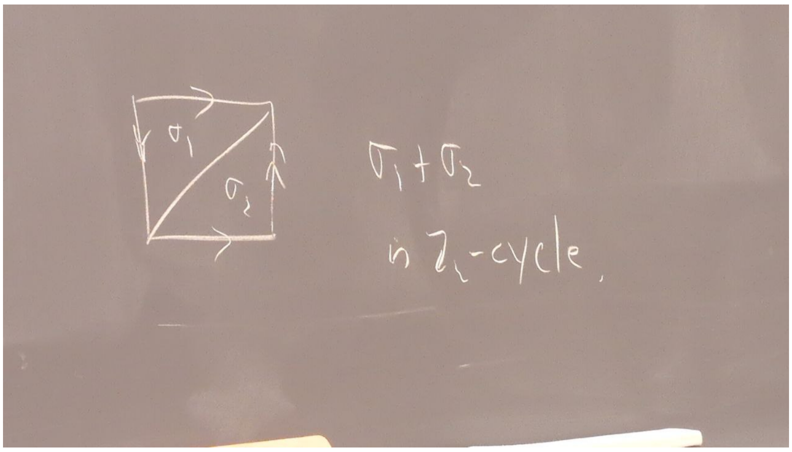
So the bottom map is indeed multiplciation by $d$.

$\square$

Let $X = \mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$.
$e^2 = \{[- : - : - : 0 : \cdots : 0]\}$.
$C_\bullet \mathbb{R}P^n = \mathbb{Z} \to \mathbb{Z} \to \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$.
Then, reducing mod 2 matrices remain the same:
$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}/2$.
$H_i(\mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0, \cdots, n; \\ 0, & \text{otherwise.} \end{cases}$
We can go from $\mathbb{Z}$ to any abelian group. Can we do this for arbitrary abelian groups?

**Theorem 70** (Universal Coefficient Theorem)**.** You can compute $H_*(X; M)$ for $H_*X$, which is universal.

Question: Why bother?
$M = \mathbb{Z}/2$: no fuss with signs, orentatio.
$\forall$ closed manifold, $H_n(M; \mathbb{Z}) = \mathbb{Z}/2$.



Also, $\beta_n(X \times Y) = \sum_{i+j=n} \beta_i X \beta_j Y$
Kunneth formula.
$M$ field eg $\mathbb{Z}_p$ or $\mathbb{Q}$ then,

$$H_n(X \times Y; M) \cong \bigoplus_{i+j=n} H_i(X; M) \otimes_M H_j(Y; M)$$

Not true for $M = \mathbb{Z}$!!!

# Borsuk-Ulam

**Theorem 71** (Borsuk-Ulam)**.** If $g : S^n \to \mathbb{R}^n$ is continuous, $\exists x \in S^n$ such that $g(x) = g(-x)$.

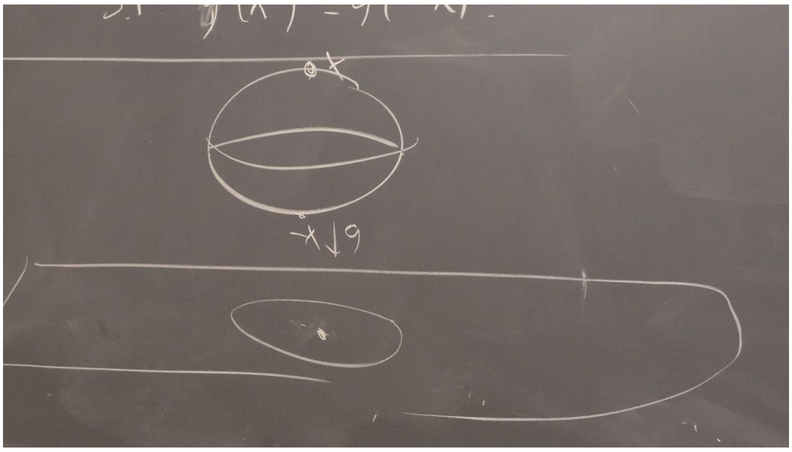


Figure 12: Here $x$ is the pole

When $n = 1$ it says $\exists$ antipodal points on equator with same temparature.
Alternate proof:

*Proof.* $f(x) = g(x) - g(-x)$ so $f(-x) = -f(x)$.
USE IVT: If $x(x) > 0, f(-x) < 0$ then $\exists x_0$ such that $f(x_0) = 0$. □

When $n = 2$ it says $\exists$ antipodal points with same temprature and humidity.

**Proposition 72** (2B6, Borsuk)**.** An odd map $f : S^n \to S^n$ has odd degree.

2B6 $\implies$ Borsuk Ulam Theorem.

*Proof.* By contradiction. Let $f(x) = \frac{g(x)-g(-x)}{\|g(x)-g(-x)\|}$.
Then $f : S^n \to S^{n-1} \hookrightarrow S^n$ is an odd map with zero degree. □

# Monday, 4/7/2025

Proof of 2B6 needs $\left\{\begin{matrix}\text{Smith}\\ \text{Transfer}\end{matrix}\right\}$ exact sequence.
Suppose $\widetilde{X} \xrightarrow{p} X$ is a double cover. Then we have SES of abelian groups:

$$0 \to \mathbb{Z}_2 \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\begin{bmatrix}1 & 1\end{bmatrix}} \mathbb{Z}_2 \to 0$$

We have SES of chain complexes:

$$0 \to S_\bullet(X;\mathbb{Z}_2) \xrightarrow{\tau} S_\bullet(\widetilde{X};\mathbb{Z}_2) \xrightarrow{p_\#} S_\bullet(X;\mathbb{Z}_2) \to 0$$

$$\begin{tikzcd} & \widetilde{X} \arrow[d] \\ \Delta^n \arrow[ur, "\widetilde{\sigma}_1{,}\widetilde{\sigma}_2"] \arrow[r, "\sigma"] & X \end{tikzcd}, \ \tau(\sigma) = \widetilde{\sigma}_1 + \widetilde{\sigma}_2$$

$$\begin{tikzcd} & (\overline{X}, \overline{x}_0) \arrow[d] \\ (Y, y_0) \arrow[ur, dotted, "\exists"] \arrow[r] & (X, x_0) \end{tikzcd} \iff \begin{tikzcd} & \pi_1(\overline{X}, \overline{x}_0) \arrow[d] \\ \pi_1(Y, y_0) \arrow[ur, dotted, "\exists"] \arrow[r] & \pi_1(X, x_0) \end{tikzcd}$$

Zig-Zag lemma $\implies$ LES:

$$\cdots \to H_i(X;\mathbb{Z}_2) \xrightarrow{\tau_*} H_i(\widetilde{X};\mathbb{Z}_2) \to H_i(X;\mathbb{Z}_2) \xrightarrow{\partial} H_{i-1}(X;\mathbb{Z}_2) \to \cdots$$

Natural w.r.t. maps of double covers, so commutative diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & X \arrow[d] \\ \widetilde{Y} \arrow[r] & Y \end{tikzcd}$$

Smith ES for $\widetilde{X} \to X \to$ Smith ES for $\widetilde{Y} \to Y$.
Recall 2B6: Odd $f : S^n \to S^n$ has odd degree.

*Proof of 2B6.* Let $P := \mathbb{R}P^n$ then odd $f : S^n \to S^n$ gives us $\overline{f} : P \to P$ given by $\overline{f}[\pm x] = [f(x)]$. This is well defined since $[f(x)] = [f(-x)]$.
We have Smith Exact Sequence:

$$0 \to H_n(P;\mathbb{Z}_2) \xrightarrow{\approx} H_n(S^n;\mathbb{Z}_2) \xrightarrow{0} H_n(P;\mathbb{Z}_2) \xrightarrow{\approx} H_{n-1}(P;\mathbb{Z}_2) \to \cdots$$

$$\cdots \to H_i(P;\mathbb{Z}_2) \xrightarrow{\partial,\approx} H_{i-1}(P;\mathbb{Z}_2) \to \cdots$$

$$\cdots \to H_1(P;\mathbb{Z}_2) \xrightarrow{\approx} H_0(P;\mathbb{Z}_2) \xrightarrow{0} H_0(S^n;\mathbb{Z}_2) \xrightarrow{\approx} H_0(P;\mathbb{Z}_2) \to 0$$

So, $H_n(P;\mathbb{Z}_2) \xrightarrow{\approx} H_n(S^n,\mathbb{Z}_2)$.
Inductively, $\overline{f}_*$ isomorphism $\implies$ $f_*$ isomorphism, lemma 249 $\implies$ $\deg f \not\equiv 0 \mod 2 \implies \deg f$ odd.

$\square$

**Theorem 73** (Ham Sandwich Theorem)**.** You can cut a sandwich with three items (ham, cheese and bread) in half.

*Proof.* Let $\underline{b} \in S^n$ and define $p_{\underline{b}} : \mathbb{R}^n \to \mathbb{R}$ by:

$$p_b(x_1, \cdots, x_n) = b_0 + b_1 x_1 + \cdots + b_n x_n$$

Then $p_b = 0$ denotes an affine hyperplane [the knife].
HST: let $F_1, \cdots, F_n \subset \mathbb{R}^n$ have finite volume. Then $\exists \underline{a} \in S^n$ such that $\forall i$,

$$\mathrm{Vol}(F_i \cap (P_a < 0)) = \mathrm{Vol}(F_i \cap (P_a > 0))$$

Where $(P_a < 0) = P_a^{-1}(-\infty, 0)$.
We start the proof now.
$f : (f_1, \cdots, f_n) : S^n \to \mathbb{R}^n$.
$f_i(\underline{b}) = \mathrm{Vol}(F_i \cap P_{\underline{b}} < 0) - \mathrm{Vol}(F_i \cap P_{\underline{b}} > 0)$.
Finite volume $\implies f(-\underline{b}) = -f(\underline{b})$ so $f$ is odd. Thus $\exists \underline{a} \in S^n$ such that $f(\underline{a}) = f(-\underline{a})$.
Then $P_{\underline{a}} = 0$ cuts sandwich in half. $\square$

# Wednesday, 4/9/2025

**Proposition 74** (2B1). a) If $D \subset S^n$ and $\exists h : D \to B(1, \mathbb{R}^k)$ homeomorphism implies,

$$\widetilde{H}_i(S^n - D) = 0 \, \forall i$$

b) If $S \subset S^n$ and $\exists h : S \to S^k \subset \mathbb{R}^n$ homeomorphism implies,

$$\widetilde{H}_i(S^n - S) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n - k - 1; \\ 0, & \text{otherwise.} \end{cases}$$

## Manifolds

**Definition.** A manifold of dimension $n$ is a topological space $X$ so that,

1) Hausdorff
2) 'Locally Euclidean' i.e. every point $x \in X$ has a neighborhood $U$, $x \in U$ and $\exists$ homeomorphism $h : U \to W \underset{\text{open}}{\subset} \mathbb{R}^n$.

eg sphere, torus, klein bottle.
Lets go back to 2B1. We want to prove Jordan Curve Theorem, which is: $f : S^1 \to \mathbb{R}^2$ has two path connected commponents.
We want $n - k - 1 = 0$ in 2B1b. Recall $\mathrm{rank}(\widetilde{H}_0(X)) = \#$ path components of $X - 1$ which gives us the result.
$\mathbb{R}^n$ is locally path connected $\implies$ path component = component.

**Definition.** $X$ is disconnected if $X = U_1 \cup U_2, U_i$ open, $U_1 \cap U_2 = \varnothing$.

## Invariance of Domain

**Theorem 75** (2B3). Let $X \subset \mathbb{R}^n$. Suppose $\exists$ homeomorphism $X \xrightarrow[\cong]{h} U$ where $U \subset \mathbb{R}^n$ is open. Then $X \subset \mathbb{R}^n$ is open.

*Proof.* Since $\mathbb{R}^n \underset{\text{open}}{\subset} S^n$ we can replace $\mathbb{R}^n$ with $S^n$.
Let $B(h(x), \epsilon)$ be a closed ball of radius $\epsilon$ small enough so that $B(h(x), \epsilon) \subset U$.
Let $D$ be the preimage. $D \coloneqq h^{-1}(B)$. $D$ is closed in $X$.
Let $S = h^{-1}(\partial B) = h^{-1}(S(h(x), \epsilon))$.
Now we use 2B1. $h\big|_D : D \to B$ implies $\widetilde{H}_0(S^n - D) = 0$. Thus $S^n - D$ is path connected.
Similarly, replacing $D$ with $S$,
$\widetilde{H}_0(S^n - S) = \mathbb{Z} \implies H_0(S^n - S) = \mathbb{Z} \oplus \mathbb{Z}$.

Thus, $S^n - S$ has exactly two path components. One of them is $S^n - D$.
Therefore, $D \setminus S$ is connected.
Thus, $S^n \setminus S = (S^n - D) \sqcup (D - S)$.
$S^n - S = C_1 \sqcup C_2$, $S^n - D \subset C_1, D - S \subset C_2$.
$S^n - D$ and $D - S$ are open components of $S^n - S$.
Thus $x \in D - S \underset{\text{open}}{\subset} S^n$. $\square$

**Corollary 76.** If $M$ is a compact $n$-dimensional manifold and $N$ is any $n$-dimensional manifold and $e : M \to N$ is an embedding (1-1 continuous), then $e$ is a homeomorphism.

*Proof.* $e(M)$ is closed [$M, N$ hausdorff, $M$ compact]. Enough to show $e(M)$ open.
$e(M)$ being open follows from Invariance of Domain. We can replace $N$ with $\mathbb{R}^n$ since $N$ is locally $\mathbb{R}^n$ and openness is a local property. $\square$

For example we can say $T^2 \not\cong S^2$.
We have another application.

**Theorem 77.** $\mathbb{R}$ and $\mathbb{C}$ are the only commutative unital finite dimensional division algebras over $\mathbb{R}$.
Equivalently, $\mathbb{R}$ and $\mathbb{C}$ are the only fields $F$ such that $F \supset \mathbb{R}, [F : \mathbb{R}] < \infty$.

This is a special case of a more famous theorem [Hopf invariant 1 problem].

**Theorem 78** (Hopf Invariant 1)**.** $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the only finite dimensional division algebras over $\mathbb{R}$.

*Proof.* We prove the easy case.
Suppose we have a division algebra $(\mathbb{R}^n, +, \cdot)$ where $n > 2$.
Define $f : S^{n-1} \to S^{n-1}$ by $f(x) = \frac{x^2}{|x^2|}$. Being division algebra gurantees we don't divide by 0.
$f(x) = f(-x) \implies$ we have a map $f : \mathbb{R}P^{n-} = S^{n-1}/\pm 1 \to S^{n-1}$.
Calculation shows that $f : \mathbb{R}P^{n-1} \to S^{n-1}$ is injective.
Not possible for $n > 2$. $\square$

# Thursday, 4/11/2025

Skipped

# Monday, 4/14/2025

# Cohomology

Cohomology is $H^*X$

$$\mathrm{Top}^{op} \xrightarrow{S_\bullet} \mathrm{Ch}^{op} \xrightarrow{*} \mathrm{CoCh} \xrightarrow{H^*} \mathrm{Gr}$$

<u>The Category of cochain complexes CoCh</u>
In this category, objects $C = \{C^*, \delta^*\}$. Then,

$$C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} C^3 \to \cdots$$

$\delta^{n+1} \circ \delta^n = 0 \forall n$.
Main difference: CoChain complex is increasing.
Morphisms: $f : C^\bullet \to D^\bullet$ And all the squares commute in the following:

$$\begin{tikzcd}
\cdots \arrow[r] & C^n \arrow[r] \arrow[d, "f_n"] & C^{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & \cdots \\
\cdots \arrow[r] & D^n \arrow[r] & D^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Cohomology:
$H^n(C^\bullet) = \frac{\ker \delta^n}{\operatorname{im} \delta\delta^{n-1}} = \frac{\text{cocycles}}{\text{coboundaries}}$
$H^n : \text{CoCh} \to \text{Ab}$
$H^* : \text{CoCh} \to \text{Gr}$
We have dual $* : \text{Ab}^{op} \to \text{Ab}$
$A^* = \text{Hom}(A, \mathbb{Z})$
$f : A \to B$ gives us $f^* : B^* \to A^*$.
eg we have:
$(\mathbb{Z}^n)^* \cong \mathbb{Z}^n$
$\mathbb{Q}^* = 0$
$(\mathbb{Z}/p)^* = 0$
$(\bigoplus_\infty \mathbb{Z})^* = \prod_\infty \mathbb{Z}$
We extend to $* : \text{Ch}^{op} \to \text{CoCh}$
We get CoChain complex from chain complex. So we want to talk about the dual of a chain complex $(C_\bullet)^*$. What happens in degree $n$?
$(C_\bullet)^n = (C_n)^*$.
$\partial^* = \delta$. Since we always index by domain,
$\partial^*_{n+1} = \delta^n$.
Double boundary and double coboundary are zero

$$0 = 0^* = (\partial \circ \partial)^* = \delta \circ \delta$$

We can write $C^\bullet, C^*, C^{-*}$.
$H^*X := H^*(S^\bullet X)$
$S^\bullet X := (S_\bullet X)^*$.
$f : X \to Y$
$H^n(f) : H^n(Y) \to H^n(X)$.

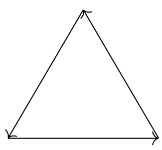
## Cohomology with Coefficients in Abelian Group $M$

We have functor $\text{Ch}^{op} \to \text{Ch}$. Instead of taking hom to $\mathbb{Z}$ we take hom to $M$. Then,

$$\text{Hom}(-, M) : \text{Ch}^{op} \to \text{CoCh}$$

$$C_\bullet \mapsto \text{Hom}(C_\bullet, M)$$

Thus, $H^*(X; M) = H^*(S^\bullet(X; M), \delta^*)$
$S^n(X; M) = \text{Hom}(S_n X, M)$
$H^n X = H^n(X; \mathbb{Z})$.
If $X$ is a CW complex we can compute $H^*X$ by using cellular methods.
$H^*X = H^*_{CW}(X)$. Consider $X = S^1 \cup_{z \mapsto z^3} D^2$. We get:



Moore space $M(\mathbb{Z}/3, 1)$
We have $X = e^0 \cup e^1 \cup e^2$ with $\partial e^2 = 3e_1$.
$C_\bullet X = \mathbb{Z}[e_2] \to \mathbb{Z}[e_1] \to \mathbb{Z}[e_0]$
$\cong \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$.
$H_* X = \{\mathbb{Z}, \mathbb{Z}/3, 0, 0, \cdots\}$
$C^\bullet X = \mathbb{Z}\widehat{e^2} \xleftarrow{\delta^1} \mathbb{Z}\widehat{e^1} \xleftarrow{\delta^0} \mathbb{Z}\widehat{e^0}$
Where $\widehat{e^2}$ etc are linear functionals. Calculating, we have:
$\mathbb{Z} \xleftarrow{3} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$
$$H^n X = \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{if } n \neq 0, 2; \\ \mathbb{Z}/3, & \text{if } n = 2. \end{cases}$$

We get the 'whole package' from homology.
$H^*(\text{pt})$
$H^*(\coprod X_\alpha)$
$H^0$
$\widetilde{H}^*$
$\widetilde{H}^*(\vee X_\alpha)$
LES of pair
homotopy invariance
excision
good pair
Mayer Vietoris
Cellular Cohomology
$H^*(\text{pt}) = \{\mathbb{Z}, 0, 0, \cdots\}$.
$H^*(\coprod X_\alpha) \cong \prod H^*(X_\alpha)$ [dual of direct sum is direct product]
$H^0 X = \prod_{\text{path components of } X} \mathbb{Z}$
Suppose we have:

$$\cdots \to S_2X \to S_1X \to S_0X \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Taking dual,

$$\cdots \leftarrow S^2X \leftarrow S^1X \leftarrow S^0X \leftarrow \mathbb{Z} \leftarrow 0$$

Cohomology is $\widetilde{H}^*X$.
Then $\widetilde{H}^*(\vee X_\alpha) = \prod \widetilde{H}^*X_\alpha$
Good pair: $H^*(X, A) = H^*(X/A, A/A) = \widetilde{H}^*(X/A)$.
LES of pair:
Recall in a pair $A \subset X$.
Similar to homology, $S^n(X, A) = \frac{S^nX}{S^nA}$
We need work to make sense of this.
Punchline: There is going to be a cohomology exact sequence:

$$\cdots \to H^n(X, A) \to H^nA \to H^nX \to H^{n+1}(X, A) \to H^{n+1}A$$

# Wednesday, 4/16/2025

Today:
LES of pair in $H^*$
Cellular cohomology
De Rham cohomology
Kronecker Pairing
UCT (Universal Coefficient Theorem).
First we talk about $S^\bullet(X; M)$. The singular $M$-cochains are duals of chains:

$$S^n(X; M) = \text{Hom}(S_nX, M) = \text{func}(\text{set of singular } n\text{-simplices } \sigma : \Delta^n \to X, M)$$

We can now define:

$$S^n(X, A; M) = \text{Hom}(S_n(X, A), M)$$

$$S_n(X, A) = S_nX/S_nA$$

We have this SES by definition of chain complexes:

$$0 \to S_\bullet A \to S_\bullet X \to S_\bullet(X, A) \to 0$$

We can use zigzag lemma to obtain a LES of homology.
Problem: Dual of SES of groups is not necessarily exact. We avoid that:

Notice that, $S_nA$ is free, therefore the SES splits. This implies, if we apply $\mathrm{Hom}(-, M)$, the SES splits. Thus, we have SES:

$$0 \leftarrow S^\bullet(A; M) \leftarrow S^\bullet(X; M) \leftarrow S^\bullet(X, A; M) \leftarrow 0$$

Applying zig-zag lemma, we have LES:

$$\cdots \to H^n(X, A; M) \xrightarrow{j^*} H^n(X; M) \xrightarrow{i^*} H^n(A; M) \xrightarrow{\delta} H^{n+1}(X, A; M) \to \cdots$$

Recall: definition of splitting: Suppose we have:

$$0 \to A \to B \underset{\pi}{\overset{s}{\leftrightarrows}} C \to 0$$

So that there is a one sided inverse $s$ such that $\pi \circ s = \mathrm{id}_C$.
$\iff$ $0 \to A \to A \oplus C \to C \to 0$ and maps are what we want.

## Cellular Cohomology

Consider CW complex $(X, \{X^n\})$. Then, cellular cochains $C^n(X; M) \coloneqq H^n(X^n, X^{n-1}; M)$.
Coboundary map:

$$\begin{tikzcd}
H^n(X^n, X^{n-1}; M) \arrow[rr, "\delta^n"] \arrow[dr, "j^*"'] & & H^{n+1}(X^{n+1}, X^n; M) \\
& H^n(X^n; M) \arrow[ur, "\delta"'] &
\end{tikzcd}$$

Then $H^*_{CW}X = H^*(C^\bullet(X; M))$.
Facts:

i) $C^\bullet X = (C_\bullet X)^*$

ii) $H^*(X; M) = H^*_{CW}(X; M)$

## DeRham Cohomology

Let $X$ be a manifold. Consider cochain complex $\Omega^\bullet X$.
$\Omega^n X$ differential $n$-forms on $X$.
Exterior derivative $d : \Omega^n X \to \Omega^{n+1} X$.
Then $\Omega^\bullet X = (\Omega^* X, d)$.
Then the DeRham Cohomology is:

$$H^n_{DR}(X; \mathbb{R}) = H^n(\Omega^\bullet X) = \frac{\text{closed } n\text{-forms}}{\text{exact } n\text{-forms}}$$

**Theorem 79** (DeRham's Theorem)**.**

$$H^n_{DR}(X) \cong H^n(X; \mathbb{R})$$

eg, we expect $H^1_{DR}(\mathbb{R}^2 - 0) \cong \mathbb{R}$.
Map: $w \mapsto \int_{S^1} w$.
Thus the generator is: $w = \frac{ydx+xdy}{x^2+y^2}$. Formally this is known as $d\theta$.
$\int_{S^1} w = 2\pi$.
Difference between closed and exact?
Let $\gamma_1$ and $\gamma_2$ be homotopic paths then $\int_{\gamma_1} w = \int_{\gamma_2} w$. But my not be true globally!
We now have:

$$H^n_{DR}(X, A) \to H^n_{DR}(X) \to H^n_{DR}(A) \to H^{n+1}_{DR}(X, A)$$

## Kronecker Pairing

Slogan: Cohomology eats Homology
Kronecker Pairing: For any space $X$ [not necessarily manifold anymore] there is a bilinear pairing:

$$\langle -, - \rangle : H^n X \to H_n X \to \mathbb{Z}$$

This is exactly the same as saying we have a map $H^n X \to (H_n X)^*$.
This is always surjective. It is a consequence of UCT.
We develop the bilinear pairing. We take cochain-chain bilinear map:
$\langle -, - \rangle : S^n X \times S_n X \to \mathbb{Z}$

$$\left\langle \alpha, \sum_i n_i \sigma_i \right\rangle = \sum_i n_i \alpha(\sigma_i)$$

Thus, in order to pass to cohomology, homology, we want to show:

$$\langle \text{cocycle, boundary} \rangle = 0$$

$$\langle \text{coboundary, cycle} \rangle = 0$$

We have the property [from the definition of $\delta (= \partial^*)$]

$$\langle \delta\alpha, c \rangle = \langle \alpha, \partial c \rangle$$

This is trivial here but in DeRham theory it is analogous to Stoke's Theorem: $\int_X dw = \int_{\partial X} w$.
Then we have:
$\langle \text{cocycle, boundary} \rangle = \langle \underset{\in \ker \delta}{\alpha}, \partial c \rangle = \langle \delta\alpha, c \rangle = 0$
$\langle \text{coboundary, cycle} \rangle = \langle \delta\alpha, \underset{\in \ker \partial}{c} \rangle = \langle \alpha, \partial c \rangle = 0$
Therefore, we naturally have:

$$\langle -, - \rangle : H^n X \times H_n X \to \mathbb{Z}$$

$\langle [\alpha], [c] \rangle = \langle \alpha, c \rangle$.

## UCT, Universal Coefficient Theorem

This gives us formula of $H^n X$ in terms of $H_n X, H_{n-1} X$.

**Theorem 80** (UCT)**.** If $H^n X$ is finitely generated $\forall n$ then there exists (split) SES:

$$0 \to (\text{torsion } H_{n-1} X) \to H^n X \to (H_n X)^* \to 0$$

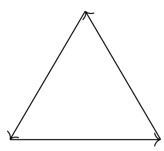
ie $H^n X \cong (H_n X)^* \oplus \text{torsion } H_{n-1} X$.

**Corollary 81.** If a space has no torsion in its homology (for example $\mathbb{C}P^n$) then $H^n X \cong (H_n X)^*$. In particular, for $\mathbb{C}P^n$,

$$H^*(\mathbb{C}P^n) = \mathbb{Z} \text{ for } 0 \leq * \leq 2n, * \text{ even}$$

Thus, $H^* \mathbb{C}P^n \cong H_* \mathbb{C}P^n$

Example: Consider $X = S^1 \cup_{z \mapsto z^3} D^2 = e^0 \cup e^1 \cup e^2$.



$C_\bullet X = \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$
$C^\bullet X = \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$
$H_* X = (\mathbb{Z}, \mathbb{Z}/3, 0, \cdots)$
$H^* X = (\mathbb{Z}, 0, \mathbb{Z}/3)$
Thus, the torsion $H_1 X \cong$ torsion $H^2 X$.
We use this to prove UCT.
We can have other coefficients. Interestingly, if our coefficients form a field, we don't have to worry about torsion. As a result,
$H^*(X; F) \xrightarrow{\cong} H_*(X, F)^*$

# Friday, 4/18/2025

## Why Cohomology is bad? Why Cohomology is good?

UCT: If $H_n X$ is f.g. $\forall n$ then,

$$H^n X \cong (H_n X)^* \oplus \text{torsion}\, H_{n-1} X$$

Furthermore, the map given by the Kronecker pairing $H^n X \xrightarrow{kp} (H_n X)^*$ given by $\alpha \mapsto \langle \alpha, - \rangle$ is onto.
The ontoness is true for any $X$ but the theorem is not.
Since dual of finite group is 0, and dual of free group is free,

$$H^n X \cong \text{Free}(H_n X) \oplus \text{torsion}(H_{n-1} X)$$

But this is not canonical: it depends on a choice of basis.
We will prove this when $X$ is a finite CW complex.

**Definition.** A chain complex $C_\bullet$ is finite free if $\forall n\, C_n$ is free and $\bigoplus_n C_n$ is f.g.
eg $C_\bullet = C_\bullet(X)$ where $X$ is a fintie CW complex.

**Definition.** An elementary chain complex is a chain complex of one of the forms:

$$0 \to 0 \to \underset{n}{\mathbb{Z}} \to 0 \to 0$$

$$0 \to 0 \to \underset{n}{\mathbb{Z}} \xrightarrow{d} \underset{n-1}{\mathbb{Z}} \to 0$$

For example, recall the triangle $C_\bullet X = \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$. This is the direct sum of:

$$0 \to 0 \to \mathbb{Z}$$

$$\oplus$$

$$\mathbb{Z} \xrightarrow{3} \mathbb{Z} \to 0$$

**Lemma 82.** Any finite free chain complex $C_\bullet$ is isomorphic to direct sum of elementary chain complexes.

*Proof.* Consider $C_2 \to C_1 \xrightarrow{\partial_1} C_0$.
Put $\partial_1$ in SNF. Choose basis $\{e_i\}$ for $C_1 \cong \mathbb{Z}^n$ and choose basis $\{f_j\}$ for $C_0 \cong \mathbb{Z}^m$ such that the matrix for $\partial_1$ is $\begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 \end{bmatrix}$.
Then $C_\bullet \cong C_n \to \cdots \to C_2 \to \text{span}\{\underbrace{e_{k+1}, \cdots, e_n}_{\ker \partial_1}\} \to 0 \oplus$ other stuff.
What is other stuff?

$\oplus(\mathbb{Z} \xrightarrow{d_i} \mathbb{Z})$
$\oplus(0 \to \mathbb{Z})$ for the $f_j$ that don't get hit at all.
We proceed by induction on the 'length' of the chain complex. $\square$

Note that UCT is true for the elementary chain complexes. Recall:

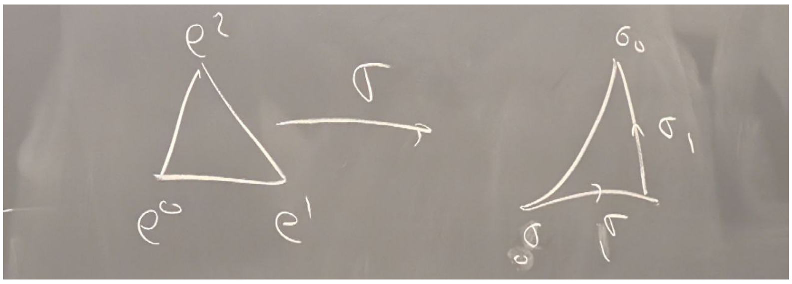
**Theorem 83** (UCT for chanin complex)**.** If $C_\bullet$ is a finite free chain complex then,

$$H^n((C_\bullet)^*) \cong (H_n C_\bullet)^* \oplus \text{torsion}\, H_{n-1}(C_\bullet)$$

The Lemma $\implies$ UCT for cc $\implies$ UCT for finite CW.
Therefore, cohomology is useless: if we know homology we can calculate the cohomology.
Now we learn why it is useful.

## Cohomology ring $H^*X$ and $f : X \to Y$ gives a ring map $f^* : H^*Y \to H^*X$

We can say $H^*X = \bigoplus_n H^n X$ is a ring. We have addition. We need multiplication.
Warm-up: $S^*X$ or $\bigoplus_n S^n X$ is a ring.
We need a multiplication. The multiplication is called the cup product.
Suppose we have $\sigma : \Delta^n \to X$ where $n = p + q$. We can break $\sigma$ to two pieces: the front $p$ face and the back $q$ face.
Front $p$-face: ${}_p\sigma : \Delta^p \to X$: append 0's at the back.
Back $q$-face: $\sigma_q : \Delta^q \to X$ : append 0's at the front.




${}_2\sigma = \sigma, \sigma_2 = \sigma$
${}_0\sigma, {}_1\sigma, \sigma_0, \sigma_1$.
Then we can define:

**Definition** (Cup Product)**.**

$$\underset{\alpha}{S^pX} \times \underset{\beta}{S^qX} \to S^{p+q}X = \mathrm{Hom}(S_{p+q}, X)$$

$$(\alpha \cup \beta)(\sigma) \coloneqq \alpha({}_1\sigma)\beta(\sigma_q)$$

$S^*X$ is a graded ring with unit.
eg $1 \in S^0X, 1(\sigma) = 1, 1\left(\sum_i n_i\sigma_i\right) = \sum_i n_i$. 1 is $\epsilon$ the augmentation map.

$(S^*X = \bigoplus_n S^nX, 0, 1, +, \cup)$ is a ring.
Graded means $(S^nX, 0, +)$ is an abelian group.
$(S^pX) \cup S^qX \subset S^{p+q}X$.

**Lemma 84** (3.6 Hatcher)**.** $\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta + (-1)^p \alpha \cup (\delta\beta)$

**Corollary 85.** 1) (coboundary) $\cup$ (cocycle) is coboundary.

2) (cocycle) $\cup$ (coboundary) is coboundary.

*Proof.* 1) $(\delta\alpha) \cup \beta = \delta\alpha \cup \beta + (-1)^p(\alpha \cup \delta\beta) = \delta(\alpha \cup \beta)$.

2) Similar.

□

**Corollary 86.** We can define $H^pX \times H^qX \to H^{p+q}X$ by $[\alpha]\cup[\beta] \mapsto [\alpha\cup\beta]$. Corollary is that this is well defined. This follows from the previous corollary.

# Monday, 4/21/2025

We continue cup product.
$H^pX \times H^qX \to H^{p+q}X$
$[\alpha] \cup [\beta] = [\alpha \cup \beta]$
$S^pX \times S^qX \to S^{p+q}X = \mathrm{Hom}(S_{p+q}X, \mathbb{Z})$
For $\sigma : \Delta^{p+q} \to X$ define:
$(\alpha \cup \beta)(\sigma) = \alpha({}_p\sigma)\beta(\sigma_q)$
Front $p$-face ${}_p\sigma : \Delta^p \to X$
${}_p\sigma(t_0, \cdots, t_p) = \sigma(t_0, \cdots, t_p, 0, \cdots, 0)$
back $q$-face $\sigma_q : \Delta^q \to X$
$\sigma_q(t_0, \cdots, t_q) = \sigma(0, \cdots, 0, t_0, \cdots, t_q)$
If $R$ is a commutative ring, then $H^*(X; R)$ and $S^*(X; R)$ is a ring.
To see $\cup$ is well-defined on $H^*$ we need to show it is independent of choice in $[\alpha], [\beta]$.
This follows from lemma 3.6 in Hatcheer: $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^p\alpha \cup \delta\beta$.

*Proof.* Apply 3-terms to $\sigma : \Delta^{p+q+1} \to X$ and compute.
$(\delta(\alpha \cup \beta))(\sigma) = (\alpha \cup \beta)(\partial\sigma) = \alpha({}_p\partial\sigma)\beta(\partial\sigma_q)$
$(\delta\alpha \cup \beta)(\sigma) = (\delta\alpha)({}_{p+1}\sigma)\beta(\sigma_q) = \alpha(\partial_{p+1}\sigma)\beta(\sigma_q)$

$$= \alpha\left(\sum_{i=0}^{p+1}(-1)^i({}_{p+1}\sigma \circ \delta^i)\right)\beta(\sigma_q)$$

$$= \sum_{i=0}^{p}(-1)^i\alpha({}_{p+1}\sigma \circ \delta^i)\beta(\sigma_q) + (-1)^{p+1}\alpha({}_p\sigma)\beta(\sigma_q)$$

Calculate everything like this and it cancels out. □

Now we talk about how a map of spaces give us a ring map on the cohomology.
If $f : X \to Y$ is continuous map of spaces we have a induced ring map $H^*(f) = f^*H^*(Y) \to H^*(X)$.
To see this, we note first that we have a induced ring map $f^\# : S^*Y \to S^*X$.
Clearly $f^\#(0) = 0$. $f^\#(\alpha + \beta) = f^\#\alpha + f^\#\beta$.
$f^\#(1_Y) =?1_Y \in S^0Y, 1_Y(\text{point}) = 1$. Thus $(f^\#1_Y)(\sigma) = 1_Y(\sigma \circ f) = 1 = 1_X(\sigma)$
Thus, $f^\#1_Y = 1_X$.
Now, $(f^\#(\alpha \cup \beta))\sigma = (\alpha \cup \beta)(\sigma \circ f) = \alpha({}_p(\sigma \circ f))\beta((\sigma \circ f)_q) = \alpha({}_p\sigma \circ f)\beta(\sigma_q \circ f) = ((f^\#\alpha)({}_p\sigma))((f^\#\beta)(\sigma_q)) = (f^\#\alpha \cup f^\#\beta)(\sigma)$.
Now we switch to the graded commutative part.
$H^*X$ graded commutative.

**Theorem 87.** $a \in H^pX, b \in H^qX \implies a \cup b = (-1)^{pq}b \cup a$.

As a consequence, if $p$ is odd, $a \in H^pX \implies a\cup a - a\cup a \implies 2(a\cup a)$. So torsion 2.
eg $a \in HH^1(S^1 \times S^1) \implies a \cup a = 0$ since $H^2(S^1 \times S^1)$ is torsion free.
DK proves this using acyclic models.
Hatcher proves it using formulas.

*Proof.* Outline of Hatcher's Proof.
Step 1: Define a chain map $\rho : S^\bullet X \to S^\bullet X$.
$\sigma : \Delta^n \to X$
$\rho(\sigma) = (-1)^{\frac{n(n+1)}{2}}\overline{\sigma}$
$\overline{\sigma}(t_0, \cdots, t_n) \coloneqq \sigma(t_n, \cdots, t_0)$
Step 2: $\rho$ and id are chain homotopic. Meaning, $\rho - \mathrm{id} = \partial P + P\partial$ where $P : S^nX \to S^{n+1}X$.

Step 3: Compute $(\alpha \cup \beta)(\rho\sigma) \pm (\beta \cup \alpha)(\sigma)$.
Then $[(\alpha \cup \beta)(\text{cocycle})] = [(\alpha \cup \beta)(\rho(\text{cocycle}))] = \pm[(\beta \cup \alpha)(\text{cocycle})]$

$\square$

Question: WHY is $H^*X$ a ring but $H_*X$ isn't?
Answer: There is no interesting map $X \times X \to X$ but the diagonal map $\Delta : X \to X \times X$ given by $\Delta(x) = (x, x)$ is interesting.

$$\begin{tikzcd} H^*(X \times X) \arrow[r, "\Delta^*"] & H^*X \\ H^*X \otimes H^*X \arrow[u] \arrow[ur, "\cup"] & \end{tikzcd}$$

Consequently, for topological groups homology have ring structure.

**Remark.** If $X$ is CW eg $X = \Delta^n$ then $\text{X} \xrightarrow{\Delta} X \times X$ is not cellular but $\simeq$ to a cellular map. 'diagonal approximation' $\leftrightarrow_p \sigma$ and $\sigma_q$.

# Wednesday, 4/23/2025

Cohomology of Sphere is not interesting.
$H^*(S^n) = \left\{ \underset{1}{\mathbb{Z}}, 0, \cdots, 0, \underset{\mu}{\mathbb{Z}}, 0, \cdots \right\}$
$1 \cup \mu = \mu$
$H^*(S^n) = \frac{\mathbb{Z}[\mu]}{\mu^2}$
Lets move on to torus.
$T^2 = S^1 \times S^1$
$H_*T = \{\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}, 0, \cdots\}$
$H^*T = \{\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}, 0, \cdots\}$
$a \cup a = 0$ so only mystery is $(1, 0) \cup (0, 1)$ in a sense.
We take the $\Delta$-complex.
$H_*T = \{\mathbb{Z}, \underset{[x]}{\mathbb{Z}} \oplus \underset{[y]}{\mathbb{Z}}, \underset{L-U}{\mathbb{Z}}, 0, \cdots\}$
$H^*T = \left\{ \mathbb{Z}, \underset{a}{\mathbb{Z}} \oplus \underset{b}{\mathbb{Z}}, o \right\}$
$a[x] = \langle a, [x] \rangle = 1$
$\langle a, [y] \rangle = 0$
$\langle b, [x] \rangle = 0$
$\langle b, [y] \rangle = 1$
$(\alpha \cup \beta)(U) = \alpha({}_1U)\beta(U_1) = \alpha(y)\beta(x) = 0 \cdot 0 = 0.$
$(\alpha \cup \beta)(L) = \alpha({}_1L)\beta(L_1) = \alpha(x)\beta(y) = 1 \cdot 1 \equiv 1$
$H^*T^2 = \frac{\mathbb{Z}[a,b]}{a^2, b^2, ab+ba} = \Lambda_{\mathbb{Z}}(a, b)$ exterior algebra.
$a \cup b = [\alpha] \cup [\beta] = [\gamma]$
Application 1: $T^2 \not\simeq S^1 \vee S^1 \vee S^2$. Idea: the right one has no natural coop produts.
Application 2: $T^2 \to S^2$ induces map on $\widetilde{H}^*, \widetilde{H}_*$
$f : H^*(T^2) \to H^*(S^2)$
$f^*(w) = f^*(a \cup b) = f^*(a)B))$
can't keep up
Lemma:
i: $H^*(X \coprod Y) = H^*X \times H^*Y$ ring product
ii: $H^*(X \vee Y) = \frac{H^*X \times H^*Y}{1_X - 1_Y}$
iii $c \in H^iX \hookrightarrow H^i(X \vee Y), d \in H^jY \hookrightarrow 0$. Then $c \cdot d = 0$
Then $c \cdot d = 0$

*Proof.* i: $\widetilde{H}^*(X \vee Y) \underset{\text{group}}{=} \widetilde{H}^*X \oplus \widetilde{H}^*Y$ $\square$

Sorry LMAO

## Friday, 4/25/2025

We continue on with the 2-torus.
$H^*T^2 = \{\underset{1}{\mathbb{Z}}, \underset{a}{\mathbb{Z}} \oplus \underset{b}{\mathbb{Z}}, \underset{a\cup b=\mu}{\mathbb{Z}}, 0, \cdots\}$
$H_*T^2 = \{\underset{pt}{\mathbb{Z}}, \underset{x}{\mathbb{Z}} \oplus \underset{y}{\mathbb{Z}}, \underset{[T^2]}{\mathbb{Z}}, 0, \cdots\}$
We want to make sure, that by kronecker pairing, $x, y$ is the dual basis of $a, b$.
We also have: $\langle \mu, [T^2] \rangle = 1$ by kronecker pairing. $\mu[T^2] = 1$.
Geometrically: setting the torus as $T^2 = S^1 \times S^1$ given by $(e^{i2\pi\theta}, e^{i2\pi\lambda})$ and $a = d\theta, b = d\lambda$. The cup product is the wedge product of these two.
$\langle a, x \rangle = \int_x a$.

**Corollary 88.** $T^2 \not\simeq S^1 \vee S^1 \vee S^2$ even though the have same $H_*, H^*_{\text{grp}}$.
They have different cohomology rings.

We try to make it a little bit more precise.

**Lemma 89.** i) $H^*(X \coprod Y) \to H^*X \times H^*Y$ is isomorphism. This is in fact a cohomology ring isomorphism.

ii) If $x_0 \in X$ and $y_0 \in Y$ have contractible neighborhoods (non-degenerate base points) then we have a map induced by the quotient map $H^*(X \vee Y) \to H^*(X \coprod Y)$ is a ring map and a group isomorphism in degrees $* > 0$.

*Proof.* i) Clear

ii) Clear from MVES (which we can apply from contractible neighborhood) □

$X \leftrightarrow X \vee Y \leftrightarrow Y$, both retracts.
Claim: suppose $c \in H^iX, d \in H^jY, i, j > 0$. Then, $i_X^*c \cup i_Y^*d = 0$. Thus $S^1 \vee S^1 \vee S^2$ has no nontrivial cup products [we pass to the product, and $(c, 0) \cup (0, d) = (0, 0)$].

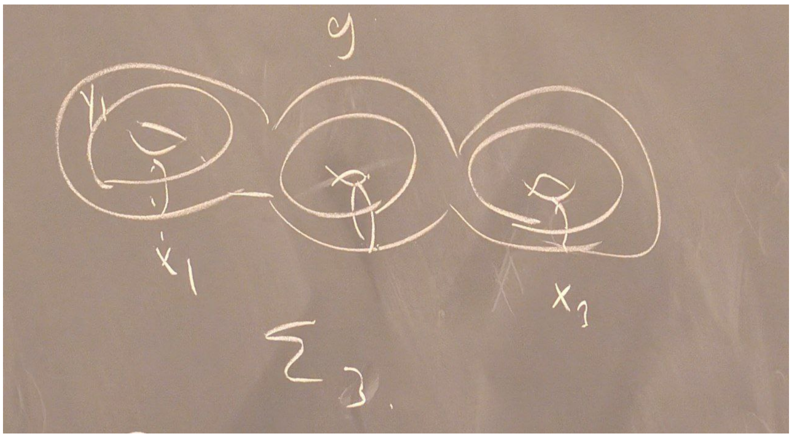
**Corollary 90.** Any map $f : S^2 \to T^2$ is trivial on $\widetilde{H}^*$ [and by UCT on $\widetilde{H}_*$]

*Proof.* $f^*\mu = f^*(a \cup b) = f^*(a) \cup f^*(b) = 0 \cup 0 = 0$ so it is trivial on $H^2$.
Trivial on $H^1$ since it is 0.
UCT $\implies$ homology is dual to cohomology. Also true for maps: $H_2S^2 \to H_2T^2$ and $H^2T^2 \leftarrow H^2T^2$ are dual. Dual of 0 map is 0 map. □

Note: there exists a nontrivial map $T^2 \to S^2$. We quotient by the boundary: $T^2 \to (T^2)/(T^2)^1 = S^2$. In fact this is an isomorphism.
This is called the degree one collapse map. Take 2-manifold, take a little disk $D^2$ inside, take $D^2/\partial D^2 \cong S^2$. This is $H_2$.
Now, let $\Sigma_g$ be the closed surface of genus $g$. In other words:

$$\Sigma_g = \underbrace{T^2 \# \cdots \# T^2}_{g\text{-torii}}$$

Question: $H^*(\Sigma_g) =$?
Note: $H_*\Sigma_g = \{\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}, 0, \cdots\}$
Thus $H^*\Sigma_g = \{\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}, 0, \cdots\}$ by UCT. We don't have the ring structure.
$H_1(\Sigma_g) = \underset{x_1}{\mathbb{Z}} \oplus \underset{y_1}{\mathbb{Z}} \oplus \cdots \oplus \underset{x_g}{\mathbb{Z}} \oplus \underset{y_g}{\mathbb{Z}}$.




take dual basis: $H^1(\Sigma_g) = \underset{a_1}{\mathbb{Z}} \oplus \underset{b_1}{\mathbb{Z}} \oplus \cdots \oplus \underset{a_g}{\mathbb{Z}} \oplus \underset{b_g}{\mathbb{Z}}$

**Corollary 91.** $a_1 \cup b_1 = \cdots = a_g \cup b_g$ and the answer is a generator of $H^2$ [say $\mu$]
$a_i \cup a_j = 0, b_i \cup b_j = 0, i \neq j \implies a_i \cup b_j = 0$

Then the matrix of bilinear product is given by:

$$\begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix}$$

$$\begin{tikzcd} \Sigma_g \arrow[r, "\psi"] & T^2 \vee \cdots \vee T^2 \\ & T_2 \coprod \cdots \coprod T^2 \arrow[u, "\phi"] \end{tikzcd}$$

We collapse the joining ring between torii to get $\psi$. The result is:

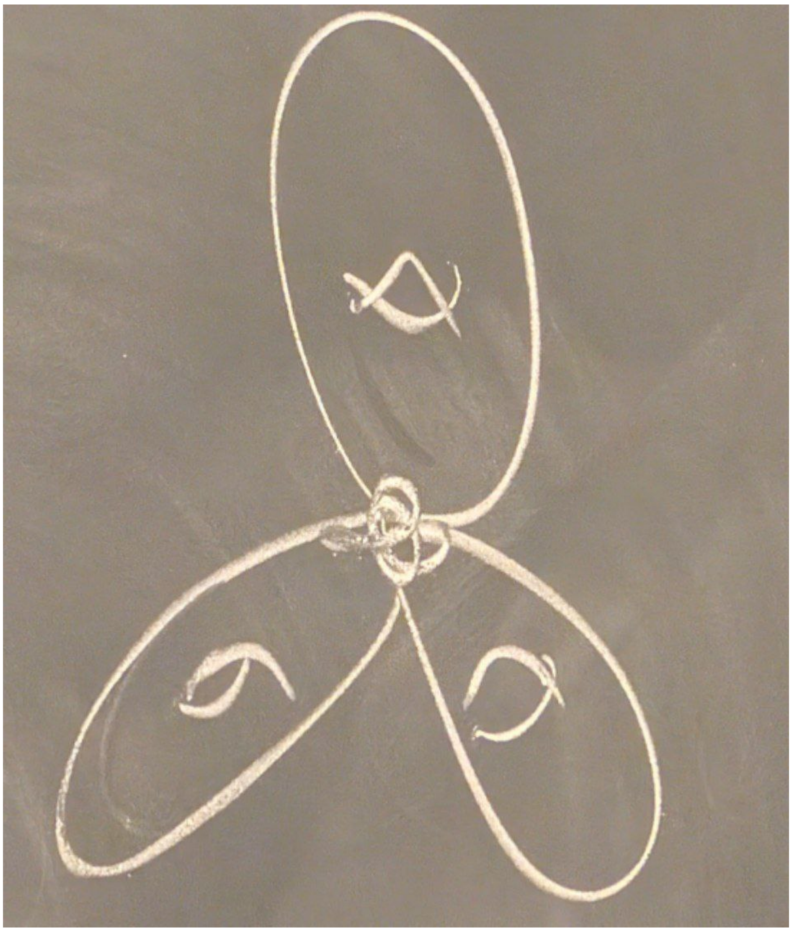


Figure 13: $\psi(\Sigma_g)$

MV $\implies \psi, \phi$ give hisomorphism on $H_1, H^1$.
MV $\implies H^2(\phi)$ isomorphism.
We have the map:

$$H^2(\Sigma_g) \leftarrow H^2(T^2 \vee \cdots \vee T^2)$$

$$\mathbb{Z} \xleftarrow{[1\cdots 1]} \mathbb{Z}^g$$

Suppose basis of $H^2(\Sigma_g)$ is $a_i \cup b_i$, basis of $H^2(T^2 \vee \cdots \vee T^2)$ be $a_i' \cup b_i'$ and $H_2(T^2 \coprod \cdots \coprod T^2)$ be $a_i'' \cup b_i''$.
Basis of $\mathbb{Z}^g$ is given by $a_i'' \cup b_i''$.
Then we have: since $H^2$ is contravariant: $\mu \leftarrow\!\shortmid a_i' \cup b_i' \mapsto a_i'' \cup b_i''$

# Monday, 4/28/2025

Other computations of $H^*T$
Künneth Theorem Poincaré duality
If $A^*$ and $B^*$ are graded commutative rings, then so is $A^* \otimes B^*$.
Also, $(A^* \otimes B^*)^n = \bigoplus_{i+j=n} A^i \otimes B^j$.
$(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'$, extended linearly to the whole ring.
We have map of graded commutative ring:

$$H^*X \otimes H^*Y \to H^*(X \times Y)$$

$$a \otimes b \mapsto P_X^*(a) \cup P_Y^*(b)$$

We like to think about this as an isomorphism (even though it's not).

**Theorem 92** (Künneth Theorem)**.** If $X$ and $Y$ are finite CW complexes,

- the above map is injective
- isomorphism if $H^iX, H^jY$ are free for all $i, j$.

$H^*(S^1 \times S^1) \cong H^*S^1 \otimes H^*S^1 = \frac{\mathbb{Z}[a]}{a^2} \otimes \frac{\mathbb{Z}[b]}{b^2} = \Lambda(a, b)$
Recall $H^*(X; R)$ is a ring as long as $R$ is a commutative ring. In the case $R = \mathbb{Q}$, the betty numbers:

$$\beta_n(X \times Y) = \sum_{i+j=n} \beta_i(X)\beta_j(Y)$$

Where $\beta_i X = \dim_{\mathbb{Q}} H_i(X; \mathbb{Q}) = \operatorname{rank} H_i X$

## Poincaré Duality

**Definition.** $M$ is a (topological) $n$-manifold if $\forall p \in M, \exists$ neighborhood $U \cong \mathbb{R}^n$.
eg $T^2$ is a 2-manifold.
We assume $M$ is connected and compact. This is sometimes called closed.

Let $A, B, C$ be $R$-modules.

**Definition** (Perfect Pairing)**.** A perfect pairing is a bilienar map $\beta : A \times B \to C$ such that the corresponding adjoint maps $A \to \operatorname{Hom}_R(B, C)$ and $B \to \operatorname{Hom}_R(A, C)$ are isomorphisms.
Note: obviously, $a \mapsto \beta(a, -), b \mapsto \beta(-, b)$.

In algtop we traditionally write $\mathbb{F}_2 = \mathbb{Z}/2$. There are two duality theorems: one with $\mathbb{F}_2$ and one with $\mathbb{Z}$.

**Theorem 93** (Poincaré Duality)**.**

i) $H_n(M, \mathbb{F}_2) = \mathbb{F}_2$. Call generator 'fundamental class' which we write as $[M] = H_n(M; \mathbb{F}_2)$. It is the 'sum' of all top dimensional simplices.

ii) 'Intersection Pairing' is a perfect pairing: $I : H^i(M; \mathbb{F}_2) \times H^{n-i}(M; \mathbb{F}_2) \to \mathbb{F}_2$ where $I(a, b) = \langle a \cup b, [M] \rangle$.

iii) $H_i(M; \mathbb{F}_2) \cong H^{n-i}(M, \mathbb{F}_2)$

iv) $H_i(M, \mathbb{F}_2) \cong H_{n-i}(M, \mathbb{F}_2)$.

Note: iii $\iff$ iv by UCT since $\mathbb{F}_2$ is a field.

**Corollary 94.** Corollary to ii: $H^*(T^2; \mathbb{F}_2) = \frac{\mathbb{F}_2[a,b]}{(a^2=0, b^2=0)}$.

**Theorem 95.** $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[a], |a| = 1$
$H^*(\mathbb{R}P^n; \mathbb{F}_2) = \frac{\mathbb{F}_2[a]}{a^{n+1}}$ [truncated polynomial ring]
Recall: $\mathbb{R}P^\infty = e^0 \cup e^1 \cup e^2 \cup \cdots$

*Proof.* Poincaré Duality and induction on $n$. We use ii.
True for $n = 1$ since $\mathbb{R}P^1 = S^1$.
Assume true for $n - 1$.
$\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n$.
Thus $i^* : H^j(\mathbb{R}P^n; \mathbb{F}_2) \to H^j(\mathbb{R}P^{n-1}; \mathbb{F}_2)$.
Claim: this is an isomorphism for $j < n$.
Proof: $C_\bullet(\mathbb{R}P^{n-1}; \mathbb{F}_2) = \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \to \cdots$, same for $\mathbb{R}P^n$. So, the chain complexes are the same up to $n - 1$.

$$\begin{tikzcd}
 & \mathbb{F}_2 \arrow[r, "0"] \arrow[d, "\cong"] & \mathbb{F}_2 \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathbb{F}_2 \arrow[d, "\cong"] \\
\mathbb{F}_2 \arrow[r, "0"] & \mathbb{F}_2 \arrow[r, "0"] & \mathbb{F}_2 \arrow[r] & \cdots \arrow[r] & \mathbb{F}_2
\end{tikzcd}$$

$0 \neq a \in H^i(\mathbb{R}P^n; \mathbb{F}_2)$
$a^{n-1} \neq 0$ by inc$^*$ and induction.
$a^n \neq 0$ by ii P.D.
$I : H^1(\mathbb{R}P^n; \mathbb{F}_2) \times H^{n-1}(\mathbb{R}P^n; \mathbb{F}_2) \to \mathbb{F}^2$ by $(a, a^{n-1}) \neq 0$ $\square$

Corollary: $\nexists r : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$ which is isomorphism on $\pi_1$.

*Proof.* Assume $r$ exists. Then $r$ is isomorphism on $\pi_1$.
Then $r_*$ is isomorphism on $H_1(; \mathbb{F}_2)$, abelianization
Then $r^*$ is iso on $H^1(; \mathbb{F}_2)$, dual
$0 = r^*(0) = r^*(a^n) = r^*(a)^n \neq 0$. Contradiction.
$\square$

See eimilarity: any $S^2 \to T^2$ is trivial on $\widetilde{H}_*$.

## Friday, 5/2/2025

Exam: Friday May 9
$X$ CW.
$A \subset X$ such that $A \cap$ cell is empty or point $\implies$ $A$ discrete.