MAGNTS

Thanic Nur Samin

November 21, 2025

Saturday, 3/29/2025

The algebraic theory of diff eqs by Bjorn Poonen

Review of linear DEs

Existence and uniqueness theorem for linear ODE:

Theorem 1. Let $\mathcal{U}$ be a simply connected subset of $\mathbb{C}$. Let $\mathcal{O}(\mathcal{U})$ be ring of holomorphic functions $\mathcal{U} \to \mathbb{C}$
$a \in \mathcal{O}(\mathcal{U}), u \in \mathcal{U}, b \in \mathbb{C}$.
Then, $\exists! f \in \mathcal{O}(\mathcal{U})$ such that $f' = af$ and $f(u) = b$.

We also have a version for system (n-tuple of functions). In that case, we change it to:
$A \in M_n(\mathcal{O}(\mathcal{U}))$
$\exists! f \in \mathcal{O}(\mathcal{U})^n$ such that $f' = Af$ and $f(u) = b$.

Remark.

- Can replace holomorphic with $C^\infty, \mathbb{C}$ with $\mathbb{R}$
- Not algebraic: solutions to $f' = z^2 f, f(0) = 1$. There exists unique solution: $f(z) = e^{z^2/3}$. But it is not a polynomial.
- $\exists$ nonlinear version but solutions only in a small neighborhood of u. eg $f' = f^2$ with $f(0) = 1$ on $\mathbb{C}$, solution is $\frac{1}{1-z}$.
- 'Simply Connected' is necessarry. Consider $f' = \frac{1}{2z} f$. Seperating variables: $d(\log f) = \frac{1}{2} d(\log z)$. Has nonzero solution on any simply connected $\mathcal{U} \subset \mathbb{C}^\times$ [a branch of $\sqrt{z}$, eg] but no non-zero holomorphic solution on $\mathbb{C}^\times$.
- We can do higher-order differential equations. We convert it into systems of first order differential equation. eg a branch of $\log z$ on an open subset of $\mathbb{C}^\times$ is a solution to $(zf')' = 0$ which we can rewrite as $f'' + \frac{1}{2} f' = 0$. This is second order, but we can introduce $g = f'$ to get the system $\begin{cases} f' = g \\ g' = -\dfrac{1}{2} g \end{cases}$. In other words, $\begin{bmatrix} f \\ g \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$
- PDEs: $\exists$ version for functions of ≥ 2 variables but it requires an extra 'integrability' hypothesis.

  Nonlinear Example: $F(x, y)$ on $\mathbb{C}^2$ such that $\frac{\partial F}{\partial x} = y, \frac{\partial F}{\partial y} = -x$. There is going to be no solution to this! The y derivative of $\frac{\partial F}{\partial x}$ should be equal to x derivative of $\frac{\partial F}{\partial y}$.

  Linear Example: No nonzero $f(x, y)$ satisfies $\frac{\partial f}{\partial x} = yf, \frac{\partial f}{\partial y} = -xf$. Proof: If we had a solution, on any open ball where f is non-vanishingg we can takke a branch of $\log f$ [call it F] and this would be a solution to the previous system! Only solution is the 0 function.

Local Systems

Let X be a topological space.
$\mathbb{C}^n_{X,\text{pre}}$ is the presheaf such that:

Definition. i) For every open $\mathcal{U} \subset X, \mathbb{C}^n_{X,\text{pre}}(\mathcal{U}) = \mathbb{C}^n$

ii) The restriction maps are the identity maps

Definition. $\mathbb{C}^n_X :=$ the sheafification of $\mathbb{C}^n_{X,\text{pre}}$.
$\mathbb{C}^n_X(\mathcal{U}) \equiv \{\underline{\text{locally}}$ constant functions $\mathcal{U} \to \mathbb{C}^n\}$

An automorphism of $\mathbb{C}^n_X$ as a sheaf of $\mathbb{C}$-vector spaces is given by a locally constant function $X \to \mathrm{GL}_n(\mathbb{C})$ [locally constant change of variable]
If $\phi : Y \to X$ then $\phi^{-1}\mathbb{C}^n_X = \mathbb{C}^n_Y$.

Definition. A local ystem $\mathcal{L}$ on X is basically a locally constant sheaf of finite dimensiotnal $\mathbb{C}$-vector spaces that,
$\exists (U_i)$ open covering on $X, n_i \in \mathbb{Z}_{\geq 0}, \phi_i : \mathbb{C}^{n_i}_{\mathcal{U}_i} \xrightarrow{\sim} \mathcal{L}\big|_{U_i}$
We have $\oplus, \otimes, \mathrm{Hom}$. Get rigid tensor category.

Example: $\mathcal{L} \simeq \mathbb{C}^n_X$ for some n. Call $\mathcal{L}$ constant.
Example $X = \mathbb{C}^\times$. For $\mathcal{U} \subset X$,

$$\mathcal{L}(\mathcal{U}) := \{\text{solutions to } f' = \frac{1}{2z} f\}$$

Claim: $\mathcal{L}$ is a 1-dim local system.

Proof. 1) If $\mathcal{U} \subset X$ is simple connected, $\exists$ branch of $\sqrt{z}$ on $\mathcal{U}$.

$$\mathcal{L}\Big|_{\mathcal{U}} \simeq \mathbb{C}_{\mathcal{U}} \cdot \sqrt{z}$$

2) X is covered by such $\mathcal{U}$.

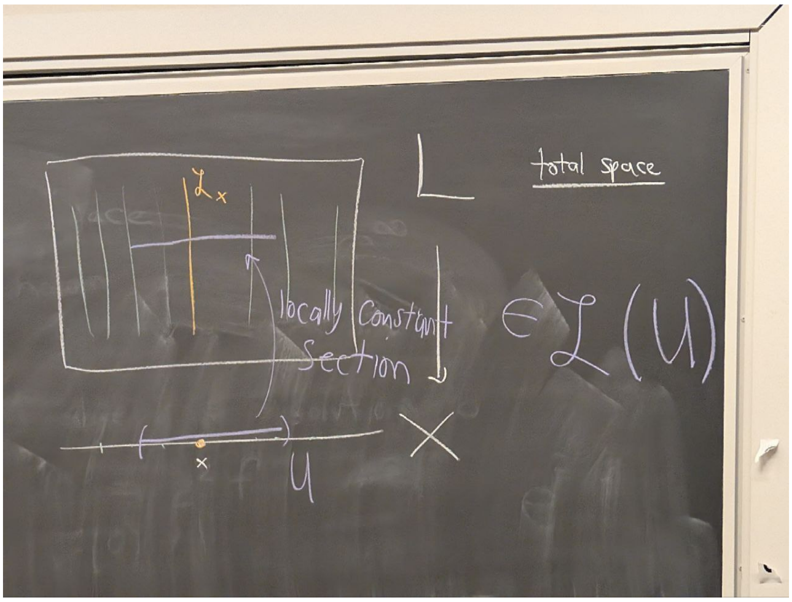
□

Claim: $\mathcal{L} \not\simeq \mathbb{C}_X$

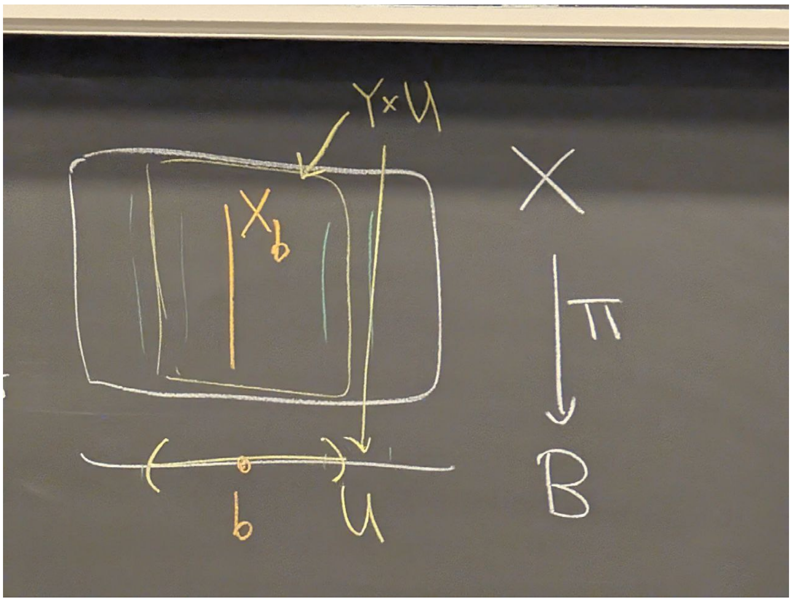
Proof. $\nexists$ solution on all of X. □

Proposition 2. For any systtem of linear ODES $f' = Af$ as in the uniquness theorem on an open subset of $\mathbb{C}$, the sheaf of solutions form an n-dim local systtem.

$\mathcal{L}$ local system. Then,
$\underline{\text{fiber of } \mathcal{L} \text{ at } x \in X} := \mathcal{L}_x$ which is n-dim $\mathbb{C}$-vector space.



Example (Relative Betti Cohomology)
X compact C^∞-manifold.
$\Gamma(X, -)$
$H^q(X, -)$
$H^q(X, \mathbb{C})$ fin dim $\mathbb{C}$-vector space.
$X \to B$ prooper submersion of C^∞ manifolds.
π_*
$R^q\pi_*$
$(R^q\pi_*)\mathbb{C}_X$ sheaf of $\mathbb{C}$-vector spaces on B.
This is a local system on B.
$(R^q\pi_*\mathbb{C}_X)_b = H^q(X, \mathbb{C})$



Question: What does a local system on $[0, 1]$ look like?

Proposition 3. Let $\mathcal{L}$ be a local system on $[0, 1]$. Then,

a) $\mathcal{L}$ is constant [proof: open cover to finite subcover, these overlap, we keep changing variables by GL_n on overlaps and get a constant function on the whole thing.]

b) $\mathcal{L}_0 \simeq \mathcal{L}_1$ [we can naturally identify fibers on 0 to fibers at 1].

Local systems and reps of π_1

$\gamma : [0, 1] \to X$ path from x to y.
If we have local system $\mathcal{L}$ on X we can pull it back to the interval. Then, $\gamma^{-1}\mathcal{L}$ is constant. Fibers of $0, 1$ [meaning fibers of x, y] gives us isomorphism $\mathcal{L}_x \simeq \mathcal{L}_y$.
If we deform the path into a homotopic one then we get the same isomorphism.
This gives us:

$$\frac{\{\text{paths from } x \text{ to } y\}}{\text{homotopy}} \times \mathcal{L}_x \to \mathcal{L}_y$$

Take $y = x$. Then our paths are loops.
We get $\pi_1(X, x)$ action on $\mathcal{L}_x$ in this case. This is a representation of this group. This is called the monodromy representation: $\mathcal{L}_x$ with action of $\pi_1(X, x)$
We also have monodromy group: $\operatorname{im}(\pi_1(X, x) \to \mathrm{GL}(\mathcal{L}_x))$
If we go back to the square root example, take $X = \mathbb{C}^\times, x = 1$
$\pi_1(\mathbb{C}^\times, 1) = \mathbb{Z}$
Then $\mathcal{L} =$ solutions to $f' = \frac{1}{2z} f$.

The we have $\pi_1(\mathbb{C}^\times, 1) \to \mathrm{GL}(\mathcal{L}_x) = \mathbb{C}^\times$
$[\gamma] \mapsto -1$
So, the monodromy group would be ± 1.

Theorem 4. Let X connected, locally simply connected topological space. Then, the functor

$$\{\text{local systems on } X\} \to \{\text{fin dim reps of } \pi_1(X, x)\}$$

$$\mathcal{L} \mapsto \mathcal{L}_x \text{ with its } \pi_1(X, x)\text{-action}$$

is an equivalence of tensor categories.

Idea of proof: Pull back the local system of the universal cover and a bunch of other stuff.

Random Walks in NT by Koukoulopoulus

- $\omega(n) = \#\{p \mid n\} = \sum_p \mathbb{1}_{p|n} \implies 2^{\omega(n)} \approx \tau(n) = \#\{d \mid n\}$
- $\log(\zeta(s)) \approx \sum_p \frac{1}{p^s}$. $e^{\log \zeta(s)} = \zeta(s)$

Think of them as RVs.

- $\{n \leq x\}$ probability space equipped with the uniform counting measure.
- Fix σ, vary $t \in [T, 2T]$ equippe wiith lebesgue measure.
  $B_p(n) = \mathbb{1}_{p|n}$ Bernoulli RV.
  $\mathbb{P}(B_p = 1) = \frac{\#\{n \leq x, p|n\}}{\#\{n \leq x\}} = \frac{\lfloor x/p \rfloor}{\lfloor x \rfloor} = \frac{1}{p} + O(1/x) \sim \frac{1}{p}$

$p^{-it} \in S^1 \rightsquigarrow\approx$ unfirom dist on S^1
$e^{-it\log p}$ is $\frac{1}{\log p}$-periodic.
$= e^{-i\{t \log p\}}$
$\mathrm{meas}(t \in [T, 2T]) = t \log p \in [a, b] (\mod 2\pi) \approx \frac{b-a}{2\pi} \cdot T$
We want to take k different primes $p_1 < \cdots < p_k$ and want to understand what is the joint distribution of $B_{p_1}, \cdots, B_{p_k}$.
$\mathbb{P}(B_{p_1} = \cdots = B_{p_k} = 1) \sim \frac{1}{p_1 \cdots p_k} \sim \prod_{j=1}^k P(B_{p_j}) = 1$.
Thus, $B_{p_1}, \cdots, B_{p_k}$ are approximately independent of each other.
$\mathbb{E}[B_p] \sim \frac{1}{p}$
$\mathrm{Var}[B_p] = \frac{1}{p} - \frac{1}{p^2}$
Thus, $E[\sum_{p \leq x} B_p] \sim \sum_{p \leq x} \frac{1}{p} \sim \log_2 x$
$\mathrm{Var}[\sum_{p \leq x} B_p] = \sum_{p \leq x} \frac{1}{p} - \frac{1}{p^2} \sim \log_2 x$

Theorem 5 (Erdös-Kac Theorem)**.**

$$\frac{\omega(n) - \log_2 x}{\sqrt{\log_2 x}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Definition. $X_n \xRightarrow{d} X \iff \forall \phi \in C_c^\infty(R), \mathbb{E}[\phi(X_n)] \to \mathbb{E}[\phi(X)] \iff \mathbb{P}(X_n \leq a) \to \mathbb{P}(X \leq a) \forall u$ that is a point of the continuity of the latter.

$$\forall u \in \mathbb{R}, \# \left\{ n \leq x : \frac{\omega(n) - \log_2 x}{\sqrt{\log_2 x}} \leq u \right\} \sim x \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-t^2/2} \, \mathrm{d}t$$

Let $p_1 < \cdots < p_k$. let $m_1, \cdots, m_k \in \mathbb{Z}$ Then,

$$\frac{1}{T} \int_T^{2T} (p_1^{it})^{m_1} \cdots (p_k^{it})^{m_k} \, \mathrm{d}t$$

$$= \frac{1}{T}\int_T^{2T} e^{it\underbrace{(m_1\log p_1+\cdots+m_k\log p_k)}_{\theta}}\,\mathrm{d}t$$

Use the fact: $\int_T^{2T} e^{i\theta t}\,\mathrm{d}t = \frac{e^{i\theta 2T}-e^{i\theta T}}{i\theta} = O(\frac{1}{|\theta|})$

$$= \begin{cases} 1, & \text{if } m_1\log p_1+\cdots+m_k\log p_k = 0; \\ \dfrac{1}{T}\dfrac{1}{|m_1\log p_1+\cdots+m_k\log p_k|}, & \text{otherwise.} \end{cases}$$

If $\exists m_i \neq 0$ then $m_1\log p_1+\cdots+m_k\log p_k = \log\frac{a}{b}$. $a,b\in\mathbb{N}$
$a,b \leq p_1^{|m_1|}\cdots p_k^{|m_k|} \leq y^{kM}$ if $|m_i|\leq M, p_i\leq y$
$|\log\frac{a}{b}| \gg \frac{1}{y^{kM}}$
$y = T^{\frac{1}{\log_2 T}}$, $k, M \leq (\log_2 T)^{1/3}$
$\mathbb{E}\left[\frac{p^{it}}{p^{\sigma}}\right] = 0, \mathrm{Var}(\frac{p^{it}}{p^{\sigma}}) = \frac{1}{p^{2\sigma}} = \begin{cases} \displaystyle\sum_p \frac{1}{p^{2\sigma}} = \infty, & \text{if } \sigma = \frac{1}{2}; \\ \displaystyle\sum_p \frac{1}{p^{2\sigma}} < \infty, & \text{if } \sigma > \frac{1}{2}; \end{cases}$
Small primes are too important, so we don't have CLT.

Theorem 6 (Selben's CLT)**.**

$$\frac{\log\zeta(\frac{1}{2}+it)}{\sqrt{\log_2 T}} \overset{d}{\Longrightarrow} N_{\mathbb{C}}(0,1) = N_{\mathbb{R}}(0,\frac{1}{2}) + iN\mathbb{R}(0,\frac{1}{2})$$

$\forall u\in\mathbb{R}$
meas $\left(t\in[T,2T] : \log|\frac{1}{2}+it| \leq u\sqrt{\frac{1}{2}\log_2 T}\right) \sim \frac{T}{\sqrt{2T}}\int_{-\infty}^{u} e^{-v^2/2}\,\mathrm{d}v$

Detour to Probability Theory

Suppose $X_1, X_2, \cdots$ are ind real RVs
$\mu_1 = \mathbb{E}[X_i], \sigma_i^2 = \mathrm{Var}[X_i]$
$s_n^2 = \sigma_1^2+\cdots+\sigma_n^2 = \mathrm{Var}(X_1,\cdots,X_n)$
Lindeberg Condition:

$$\frac{1}{s_n^2}\sum_{j=1}^{n}\mathbb{E}\left[(X_j-\mu_j)^2\cdot\mathbb{1}_{|X_j-\mu_j|>\epsilon s_n}\right]\to 0$$

Then $\frac{X_1+\cdots+X_n-\mu_1-\cdots-\mu_n}{s_n} \overset{d}{\Longrightarrow} \mathcal{N}(0,1)$
We use method of moments, see Billingsley.
Now we prove Erdös=Kac

Proof. Consider perfect random walk.
$\forall p, B_p^{\mathrm{Model}}$ is Bernoulli, $\mathbb{P}(B_p^{\mathrm{Model}}=1) = \frac{1}{p}$ i.i.d. of each other.
CLT $\Longrightarrow \frac{\sum_{p\leq y} B_p^{\mathrm{Model}} - \log\log p}{\sqrt{\log_2 y}} \overset{d}{\Longrightarrow} \mathcal{N}(0,1)$ as $y\to\infty$
$\omega(n) = \#\{p \mid n, p\leq x\} = \#\{p\mid n, p\leq x\} + O\left(\frac{\log x}{\log y}\right)$.

$$\sum_{n\leq x}\omega(n)^2 = \sum_{n\leq x}\sum_{p_1p_2|n} = \sum_{p_1p_2\leq x}\#\{n\leq x, p_1p_2\mid n\}$$

p_2, p_2 could be too large $p_1, p_2 \approx x^{\frac{2}{3}}$.
Idea: choose y cleverly in $\omega(n) = \#\{p\mid n, p\leq x\} = \#\{p\mid n, p\leq x\} + O\left(\frac{\log x}{\log y}\right)$.
We choose $y = x^{1/\log_3 x}$

$$\sum_{n\leq x}\omega(n,y)^k = \sum_{n\leq x}\sum_{p_1,\cdots,p_k\leq y, p_1,\cdots,p_k|n} 1$$

$$= \sum_{p_1,\cdots,p_k \le y} \left(\frac{x}{[p_1,\cdots,p_k]} + O(1) \right)$$

total error $y^k = o(x)$

$$\equiv x\mathbb{E}[B_p^{\text{Model}}]$$

$\square$

The algebraic theory of diff eqs by Bjorn Poonen 2

Vector Bundles

X complex manifold
$m = \dim X$
$\mathcal{O} = \mathcal{O}_X$, sheaf of holomorphic functions.

Definition. Vector bundle on $X :=$ locally free $\mathcal{O}$-modules that is locally of finite rank.

Vector Bundle	Rank
$\mathcal{O}$	1
Tangent Bundle $\mathcal{T}$	m
Sheaf of 1-forms $\Omega^1 := \text{Hom}(\mathcal{T}, O)$	m

Table 1: Rank of Vector Bundles

For each $x \in X$ we get:

- The stack $V_{(*)}$ which is a finite free $\mathcal{O}_{(*)}$ module
- Thee fiber $V_* := V \otimes_{\mathcal{O}} k_\times = V_{(*)}/m_* V_{(*)}$ which form a fin. dim $\mathbb{C}$-vector space.

Example: $\mathcal{L}$ local system, $\mathcal{V} := \mathcal{O} \otimes_{\mathbb{C}} \mathcal{L}, \mathcal{L}_* = V_*$
$\mathcal{L} =$ sheaf of locally constant functions.
$V =$ sheaf of all holomorphic function.

Derivations

A is $\mathbb{C}$-algebra.

Definition (Derivation of A)**.** :A $\mathbb{C}$-algebra.
Derivation of A: a $\mathbb{C}$-lindar map $D : A \to A$ such that $D(fg) = D(f)g + fD(g)$ for all f, $g \in A$.
eg $\frac{\partial}{\partial x}, y^2 \frac{\partial}{\partial x}$

As sheafs of $\mathbb{C}$-algebrras
Derivation D of $A =$ collection $(D_u)_{u \subset X}$ such that D_u is a derivation of $A(\mathcal{U})$ compatible with the restriction.
$\text{Der}(\mathcal{A}) = \{\text{all derivations of } \mathcal{A}\}$
$\mathscr{D}er(A) =$ the sheaf $\mathcal{U} \to \text{Der}(A\big|_u)$
For each vector field function $t \in \mathcal{T}, f \in \mathcal{O}, x \in X$
Let $(D_t f)(x) :=$ directional derivative of f in the diretion of $t(x)$
$D_t f \in \mathcal{O}$
Get $T \xrightarrow{\sim} \mathscr{D}er, t \mapsto D_t$
The pairing $\mathcal{T} \times \mathcal{O} \to \mathcal{O}$ given by $t, f \mapsto D_t f$
$\mathcal{O}$-linear in t, $\mathbb{C}$-linear in f. Get:
$\mathcal{O} \xrightarrow{d} \text{Hom}(\mathcal{T}, \mathcal{O}) =: \Omega^1$

$1 \mapsto df$.
$V \simeq \mathcal{O}_x \xrightarrow{e^{z'}} \mathcal{O}_x$
$v \mapsto 1 \leftarrow\!\shortmid e^z$
To equip V with a rule for taking directional derivatives of of sections of V we should specify a pairing
$T \times V \to V, t, v \mapsto \nabla_t v, \mathrm{Home}(T, V)$.
$\nabla : V \to \mathrm{Hom}(T, V) = \Omega^1 \otimes V$
Given (γ, δ) , each $D \in \mathscr{D}er(X) = T(X) = \mathrm{Hom}(\Omega^1, 0)$ induces:

$$\nabla_D : \mathcal{V} \xrightarrow{\nabla} \Omega^1 \otimes V \xrightarrow{D \otimes 1} \mathcal{O} \cdot V = V$$

eg d is a connection on $\mathcal{O}$
eg $\omega \in \Omega^1(X)$
$d + \omega : \mathcal{O} \to \Omega^1, f \mapsto df + f\omega$

Proposition 7. Every connection on $\mathcal{O}$ is $d + \omega$ for some $\omega \in \Omega^1(X)$

Proof. Let ∇ be a connection on $\mathcal{O}$. Then,

$$\nabla(fg) = dfg + f\nabla g$$

$$d(fg) = dfg + fdg$$

□

$(\nabla - d)(fg) = f(\nabla - d)g$
Thus, $\nabla - d$ is $\mathcal{O}$-linear hence $\mathcal{O} \to \Omega^1, f \mapsto f\omega$ for some $\omega \in \Omega^1(X)$
We also have: every connection on $\mathcal{O}^n$ is $d + \omega$ for some $\omega \in \Omega^1(X)$.
Fix (V, ∇).
$v \in V$ is called horizontal if $\Delta v = 0$.
$V^\nabla := \ker \nabla \subset V$.
subsheaf of horizontal schemes
eg $\mathcal{U} \subset \mathbb{C}, V = \mathcal{O}^n, A \in M_n(\mathcal{O}(\mathcal{U})), \nabla = d - A dz$
Then horizontal schems of V = solutions $f \in \mathcal{O}^n$ to the system $f' = Af$
eg $X = \mathbb{C}, V = \mathcal{O}, \nabla = d - z^2\,\mathrm{d}z$
$\nabla f = df - fz^2\,\mathrm{d}z$
So $\nabla f = 0 \iff f' = z^2 f$
$\gamma^\nabla = \mathbb{C}_x \cdot e^{z^3/3}$

Proposition 8. $\dim X = 1, (V, \nabla)$ on X. Then, V^∇ is a n-dim local system on X.

Curvature

∇ on V induces a seq of $\mathbb{C}$-lienar maps.

$$V \xrightarrow{\nabla} \Omega^1 \otimes_V \xrightarrow{\nabla_1} \Omega^2 \otimes V \to \cdots$$

Where $\nabla_i(\omega \otimes v) = d\omega v + (-1)^i \omega \otimes v$
The curvture of ∇ is:
$K := \nabla_1 . \nabla : V \to \Omega^2 \otimes V$
It turns out that K is $\mathcal{O}$-linear so K is a global section of

$$\mathrm{Hom}(V, \Omega^2 \otimes V) = \Omega^2 \otimes \mathrm{End}(V)$$

∇ is an integrable connection, a flat connection if and only if $K = 0$
eg If $\dim X = 1$ then $\Omega^2 = 0$ so $K = 0$ automatically.
eg If $(V, \nabla) = (0, d)$ then

$$\mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 0 > \cdots$$

is the usual de Rham complex.
eg If $(V, \nabla) = (\mathcal{O}, d + \omega)$

$K(1) = \nabla_1(\nabla 1) = \nabla_1 \omega = d\omega$
So $K = d\omega$
∇ is integrable $\iff$ ω is a closed 1-form.
eg Let $X = \mathbb{C}^2$ with coords $x, y, V = 0, \omega = -y\,\mathrm{d}x + xdy \in \Omega^1(X)$
Let $\nabla = d + \omega$ then $K = d\omega \equiv 2\mathrm{d}x \wedge \mathrm{d}y \neq 0$. ∇ is not integrable.
V^∇ is sheaf of solutions to $df + \omega f = 0$ so no nonzero solutions.
$V^\nabla = 0$

Random Walks in NT by Koukoulopoulus 3

Recall:
$\omega(n) = \sum_{p \le x} \mathbb{1}_{p|n} \approx \sum_{p \le y} \mathbb{1}_{p|n}, y = x^{1/\log_3 x}$
$\log|\zeta(\frac{1}{2} + it)| \approx \log|\zeta(\frac{1}{2} + \frac{1}{\log y} + it)| \approx \sum_{p \le y} \frac{1}{p^{\frac{1}{2}+it}}$ $y = T^{1/l \log_3 T}, t \in [T, 2T]$
$X_1, X_2, \cdots$ i.i.d. mean 0, var 1.
$\frac{X_1 + \cdots + X_n}{\sqrt{N}} \implies N(0,1)$
$f_N(\alpha) = \frac{1}{\sqrt{N}} \sum_{n \le \alpha N} X_n$, $\mathrm{Var}(f_N) = \frac{\alpha N}{N} = \alpha \implies N(0, \alpha)$
$f_N(\beta) - f_N(\alpha) = \frac{1}{\sqrt{N}} \sum_{\alpha N < n \le \beta N} X_n \implies N(0, \beta - \alpha)$.
f_N: brownian motion.

Theorem 9 (Billingsley)**.** If n simple uniform random from $[1, x] \cap \mathbb{Z}$, then the stochastic process $g : [0,1] \to \mathbb{R}$,

$$g(\alpha) = \frac{\#\{p \mid n, \log_2 p \le \alpha \log_2 x\} - \alpha \log_2 x}{\sqrt{\log_2 x}}$$

So brouwnian motion on $[0, 1]$.

What is the distribution of g if we condition on n having $r = \rho \log_2 x, \rho =$ constant. Convergene to brownian bridge = brownian motion given end propbability = 0.

Maximum of ζ

$\max_{t \in [T,2T]} \log\left|\zeta\zeta(\frac{1}{2} + it)\right| \sim \sqrt{\frac{1}{2}\log T \log_2 T} = \sqrt{\log T} \cdot \sqrt{\frac{1}{2}\log_2 T}$
Conjeture (Farner-Gortek Hughes)
$\frac{1}{T}\int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k}\,\mathrm{d}t \sim \frac{1}{2} C_k (\log T)^{k^2}$
dominated by $\log|\zeta|$
Conj of Fyodorov-Hiary–Keating about the distribution of the local maximum.
$M(t) := \max_{h \in [0,1]} \log|\zeta(\frac{1}{2} + i(t+h))|$
Dostrobution of $M(t)$ when t in uniform $[T, 2T]$
Conjecture: $M(t) = \log_2 T - \frac{3}{4}\log_3 T + O(1)$ a.s.
$\log|\frac{1}{2} + i(t+h)| \approx k \sum_{p \le T} \frac{1}{p^{1/2+it+ih}} \to S(h) = \sum_{p \le T} \mathrm{Re}\frac{X_p}{p^{1/2+ih}}$ where $X_p = \mathrm{Unit}(S^1)$
mutually independent. $p^{it} \approx \mathrm{Uniform}(S)$.
$\mathbb{E}[S(h_1)S(h_2)] = \sum_{p_1, p_2 \le T} \mathbb{E}\left[\mathrm{Re}\frac{X_{p_1}}{p^{\frac{1}{2}+ih_1}}\mathrm{Re}\frac{X_{p_2}}{p^{1/2+ih_2}}\right]$
$= \frac{1}{4}\sum_{p \le T}\left(\frac{1}{p^{1+i(h_1-h_2)}} + \frac{1}{p^{1+i(h_2-h_1)}}\right) = \begin{cases} \dfrac{1}{2}\log_2 T, & \text{if } |h_1 - h_2| \le \dfrac{1}{\log T}; \\ 0, & \text{if } |h_1 - h_2| \gg 1; \end{cases}$
Easier Problem:
$\max(Z_1, \cdots, Z_N)$
$Z_i \sim N(0, \frac{1}{2}\log_2 T, N = l \log T)$.
$\mathbb{P}\max_{1 \le i \le N} Z_i \le \mu$
$= \mathbb{P}(Z_i \le \mu)^N$
$= (1 - \mathbb{P}(Z_1 > \mu))^N$