NT Reading 2

Thanic Nur Samin

Contents

9 Thursday, 1/16/2025, Cartan Subalgebras (CSA) by Rostyslav

Corollary 1 (15.3)**.** Let L be semisimple. CSA's of L are precisely the maximal toral subalgebras of L.

Proof. $\implies$: Let H be a maximal toral subalgebra.
$\implies$ H-abelian $\implies$ H-nilpotent, $N_L(H) = H$ since $L = H + \bigsqcup_{\alpha\in\Phi} L_\alpha$ with $[H, L_\alpha] = L_\alpha$ for $\alpha \in \Phi \implies H$-CSA.
$\impliedby$: Let H-CSA. $x = x_s + x_h$ by Jordan decomposition.
$\implies L_0(\operatorname{ad} x_s) \subset L_0(\operatorname{ad} x)$ for $x \in L$ semisimple.
$L_0(\operatorname{ad} x_s) = C_L(x_s)$ since $\operatorname{ad} x$ is diagonal.
H- minimal Engel.
By defintion, $\implies L_0(\operatorname{ad} x_s) = C_L(x_s) = H$.
But $C_L(x_g)$ contains maximal toral subalgebra which is CSA. Thus it itself is minimal Engel. H-maximal toral.
Detaills:
$L \subset H$. H is CSA $\iff$ H is nilpotent, $N_L(H) = H$.
$L_0(\operatorname{ad}(x_s)) \subset L_0(\operatorname{ad}(x))$.
$L_0(\operatorname{ad}(x_s)) = \{y \in L \mid \operatorname{ad}(x_s)(y) = 0\} = \{y \in L \mid [x_s, y] = 0\}$.
$[x, y] = \underbrace{[x_s, y]}_{=0} + [x_n, y]$.
$\operatorname{ad}(x)^m(y) = \operatorname{ad}(x_n)^m(y) = 0$ for $m \gg 0$.

□

Lemma 2 (15.4.B)**.** Let $\phi : L \to L'$ [epimorphism]. Let H' be CSA of L'. Then, any CSA of $\phi^{-1}(H')$ is also a CSA of L.

Definition. $x \in L$ is callled strongly ad-nilpotent if $\exists y \in L$ and $\exists a \neq 0$ eigenvalue of $\operatorname{ad} y$ such that $x \in L_a(\operatorname{ad} y)$.

$$[L_a(\operatorname{ad} y), L_b(\operatorname{ad} y)] \subset L_{a,b}(\operatorname{ad} y)$$

Remark. By Lemma 15.1 if x is strongly ad-nilpotent then x is ad-nilpotent.
Recall: $x \in L_a(\operatorname{ad} y)$ means $\exists m > 0 : (\operatorname{ad} y - a \cdot \operatorname{id})^m(x) = 0$.
This implies that $\operatorname{ad} x$ is nilpotent after some calculation.

Definition. $\mathcal{N}(L)$ is the set of strongly ad-nilpotent elements.

Definition. $\mathscr{E}(L) < \operatorname{Int} L := \langle \exp(\operatorname{ad} x) \mid x \text{ is ad-nilpotent} \rangle$ generated by $\forall \exp \operatorname{ad} x$ where $x \in \mathcal{N}(L)$.

Remark. $\mathcal{N}(L)$ is stable under $\forall x \in \operatorname{Aut}(L)$.
Thus, $\mathscr{E}(L) \trianglelefteq \operatorname{Aut}(L)$.
$K \subset L \implies \mathcal{N}(K) \subset \mathcal{N}(L)$.
Then,

Definition. $\mathscr{E}(L, K)$ is generated by $\exp \operatorname{ad} x \forall x \in \mathscr{E}(K)$

Then, $\mathscr{E}(K) = \mathscr{E}(L, K)$.
If $\phi : L \to L'$ is an epimorphism then $\phi(L_a(\operatorname{ad} y)) = L'_a(\operatorname{ad} \phi(y))$.
$\implies \phi(\mathcal{N}(L)) = \mathcal{N}(L')$.

Lemma 3 (16.1)**.** Let $\phi : L \to L'$ be an epimorphism. If $\sigma'' \in \mathscr{E}(L') \implies \exists \sigma \in \mathscr{E}(L)$ such that:

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L' \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma'} \\ L & \xrightarrow{\phi} & L' \end{array}$$

Proof. If $\sigma' = \exp \operatorname{ad} x'$ $x' \in \mathcal{N}(L')$ $\exists x$ s.t. $\phi(x) = x'$.
$\forall z \in L, (\phi \circ \exp \operatorname{ad}_L x)(z) = \phi(z + [x, z] + [x, [x, z]]) = \phi(z) + [x', \phi(z)] + [x', [x', \phi(z)]]$
$= (\exp \operatorname{ad}_L x')(\phi(z)) \implies$ QED.

□

Theorem 4 (16.2)**.** Let L be solvable. Let H_1, H_2 be CSA's of L.
Then, H_1 is conjugate with H_2 by an element of $\mathscr{E}(L)$.

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Proof. Induction on $\dim L$.
Base case: suppose $\dim L = 1$. Since L is nilpotent L must be trivial.
Assume that L is not nilpotent. L–solvable $\implies$ L has non-zero abelian ideals. eg least non-zero term of the derived series. Choose such A of least possible dimension.
Set $L' = L/A$. We have $\phi : L \to L/A = L'$ given by $x \mapsto x'$.
Lemma 15.4(image of CSA is CSA) implies H'_1, H'_2 are CSAs of the solvable algebra L'. By induction $\exists \sigma \in \mathscr{E}(L')$ such that $\sigma(H'_1) = H'_2$.
Lemma 16.1(the commutative diagram) implies $\exists \sigma \in \mathscr{E}(L)$ such that the diagram commutes. So, σ maps $K_1 = \phi^{-1}(H'_1)$ to $\phi^{-1}(H'_2) = K_2$
But now H_2 and $\sigma(H_2)$ are both CSA"s of K_2.
If K_2 is smaller than L induction allows us to find $\tau' \in \mathscr{E}(K_2)$ such that $\tau'\sigma(H_1) = H_2$.
But $\mathscr{E}(K_2)$ consists of restrictions of $\mathscr{E}(L, K_2)$ to K_2.
$\exists \tau$ such that $\tau\sigma(H_1) = H_2$ for $\tau \in \mathscr{E}(L) \implies$ done.
Otherwise $L = K_2 = \sigma(L_1)$.
$K_2 = K_1$ and $L = H_2 + A = H_1 + A$.
Theorem 15.3 $\implies$ CSA $H_2 = L_0(\operatorname{ad} x)$ for suitable $x \in L$.
A being $\operatorname{ad} x$ stable, so by lemma 15.1,

$$A = A_0(\operatorname{ad} x) \oplus A_*(\operatorname{ad} x)$$

and each summand is stable under $H_2 + A$.
Since A is minimal, $A = A_0(\operatorname{ad} x)$ or $A = A_*(\operatorname{ad} x)$
A cannot be equal to $A_0(\operatorname{ad} x)$ since in that case $A \subset H_2, L = H_2$.
But since L is not nilpotennt we have a contradiction.
Thus, $A = A_*(\operatorname{ad} x) \implies A = L_*(\operatorname{ad} x)$. Since $L = H_1 + A$ we can write $x = y + z$ with $y \in H_1, z \in A_*(\operatorname{ad} x)$.
Since $\operatorname{ad} x$ is invertible on $L_*(\operatorname{ad} x)$ we can write $z = [x, z']$ where $z' \in L_*(\operatorname{ad} x)$.
A-abelian $\implies (\operatorname{ad} z')^2 = 0$.
Thus, $\exp \operatorname{ad} z' = 1_L + \operatorname{ad} z'$.
Applying to x we have, $x - z = y$.
$\implies H_0 = L_0(\operatorname{ad} y)$ must also be a CSA of L. Since $y \in H_1, H \supset H_1$ and both minimal Engel, $H = H_1$.
H_1 is conjuate to H_2 using $\exp \operatorname{ad} z'$.
We only need to show that, $\exp \operatorname{ad} z' \in \mathscr{E}(L)$.
z' can be written as sum of strongly ad-nilpotent elements of $A = L_*(\operatorname{ad} x)$
$\implies A$-abelian so $\exp \operatorname{ad} z' = \prod \exp \operatorname{ad} z_i \in \mathscr{E}(L)$.
So we're done.

□

Consider B = upper triangular matrices. It is a lie algebra. What is a CSA of this?
Attempt: we have H = upper triangular matrices with 0 diagonal. $N_B(H) = B$. But it is not nilpotent so it doesn't work.
However, attempt 2: we can take H = diagonal matrices.
In fact, if $\mathfrak{g} \subset \mathfrak{gl}_n$ is a subalgebra and $\mathfrak{g}$ contains a diagonal matrix with all entries different, then the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing all diagonal matrices on $\mathfrak{g}$ is a CSA.

Definition. A maximal solvable subalgebra of a lie algebra L is called a Borel subalgebra.

Lemma 5 (16.3.A)**.** If B is a borel subalgebra of L then $B = N_L(B)$. Aka, Borel subalgebras are self normalizing.

Proof. Let $x \in N_L(B)$. Then, $B + Fx$ is a subalgebra of L. It is solvable since $[B + Fx, B + Fx] \subset B$. Since B is maximal, we must have $x \in B$. □

Lemma 6 (16.3.B)**.** If $\operatorname{Rad} L \neq L$ then there is a bijection between the sets of Borel subalgebras of L and Borel subalgebras of $L/\operatorname{Rad} L$.

Proof. $\operatorname{Rad} L$ is a solvable ideal of L.
Therefore, $B + \operatorname{Rad} L$ is a solvable subalgebra of L.
$\implies$ by maximality, we're done. □

Definition. Let H be a CSA in a semisimple lie algebra L, Φ a root system of L relative to H. Fix a base Δ and a set of positive roots.
Set $B(\Delta) = H \sqcup_{\alpha > 0} L_\alpha$
And $N(\Delta) = \sqcup_{\alpha \ngtr 0} L_\alpha$.
Then $B(\Delta)$ is the standard Borel subalgebra relative to H.
$N(\Delta)$ is the derived algebra of $B(\Delta)$.

Lemma 7 (16.3.C.1)**.** $N(\Delta)$ is nilpotent.
If $x \in L_\alpha (\alpha > 0)$ then,
Application of $\operatorname{ad} x$ to root vector increases the height by at least 1.
$\implies$ decreasing central series goes to zero.
Thus, $B(\Delta)$ is solvable.
Let $K \supset B(\Delta)$. Then, K is stable under $\operatorname{ad} H$.
Then K must include some L_α with $\alpha < 0$.
Thus, simple $S_\alpha \subset K \implies K$ is not solvable.

Note: $S_\alpha = \langle L_\alpha, L_{-\alpha}, H \rangle$

Lemma 8 (16.3.C2)**.** All standard Borel subalgebras of L relative to H are conjugate under $\mathscr{E}(L)$.

Proof. By 14.3 the reflection σ_α acting on H may be extended to an inner automorphism τ_α of L which is, by construction, in $\mathscr{E}(L)$.
τ_α would send $B(\Delta)$ to $B(\sigma\Delta)$.
The Weyl Group is generated by those reflections, so we see that $\mathscr{E}(L)$ will act transitively on standard Borel subalgebras relative to H.

□

Theorem 9 (16.4)**.** The Borel subalgebras of an arbitrary Lie algebra L are all conjugate under $\mathscr{E}(L)$.

We omit the proof for now.

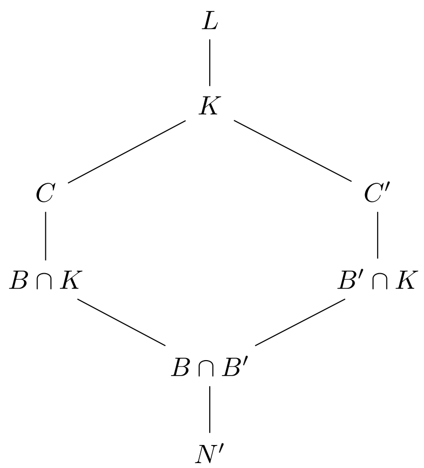
Corollary 10. All the CSAs of lie algebra L are all conjugate under $\mathscr{E}(L)$.

Proof. Let H, H' be CSAs. They're nilpotent by definition. Therefore, they're solvable.
Therefore, $H \subset B, H' \subset B'$ where B, B' are some borel subalgebra.
By the previous theorem, $\exists \sigma \in \mathscr{E}(L)$ such that $\sigma(B) = B'$.
Thus, $\sigma(H)$ and H' are CSAs of B'.
By theorem 16.2, $\exists \tau' \in \mathscr{E}(B')$ such that $\tau'\sigma(H) = H'$.
But τ' is a restriction of some $\tau \in \mathscr{E}(L, B') \subset \mathscr{E}(L)$. Therefore,

$$\tau\sigma(H) = H',\ \tau\sigma \in \mathscr{E}(L)$$

□

11 Thursday, 1/30/2025, Cartan Subalgebra (CSA) by Rostyslav



We prove theorem 16.4.

Proof. First induction hypothesis $\dim L$ upwards.
base: $\dim L = 1$ is trivial.
WLOG by lemma 16.1 and 16.3B, we can assume that L is semisimple.
Fix a standard borel subalgebra relative to some CSA.
Suffices to shwo that $\forall B'$ -other borel subalgebra is conjugate to B under $\mathscr{E}(L)$.
If $B' \cap B = B$ then $B' = B$ by maximality (both are borel).
Second induction hypothesis
$\dim(B \cap B')$ downwards for all larger dimension are conjugate.
(1) Suppose that $B \cap B' \neq 0$.
Case i: set N' of nilpotent elements of $B \cap B'$ is nonzero.

B-standard $\implies$ N' subspace derived alg of $B \cap B'$ consists of nilpotent elements.
$\implies$ N'-ideal of $B \cap B'$.
N'-not an ideal of L $\implies$ $K = N_L(N)$ is proper.
Consider the action of N' on $B/(B \cap B')$ induced by ad for all $x \in N'$ acts nilpotently on this vector space.
Theorem 33 $\implies$ $\exists y$ such that $y + (B \cap B')$ killed by $\forall x \in N$. ie st $[xy] \in B \cap B', y \notin B \cap B'$ but $[xy]$ is also $[B, B]$ $\implies$ $[xy]$ is nilpotent $\implies$ $[xy] \in N'$ or $y \in N_R(N')$.
$y \notin B \cap B' = B \cap K$
Same way $B \cap B' \subsetneq B' \cap K$.
$B \cap K, B' \cap K$ solvable subalgebra.
C, C' borel subalgebra containing them $K \neq L$ by induction $\exists \sigma \in \mathscr{E}(L, K) \subset \mathscr{E}(L)$ such that $\sigma(C') = C$.
Since $B \cap B \subsetneq C$ and $B \cap B' \subseteq C'$ second induction hypothesis implies $\exists \tau \in \mathscr{E}(L)$ such that $\tau\sigma(C') \subset B$
$B \cap \tau\sigma(B') \supset \tau\sigma(C') \cap \tau\sigma(B') \supset \tau\sigma(B' \cap K) \supsetneq \tau\sigma(B \cap B')$
$\implies$ Second induction hypothesis
B is conjugate under $\mathscr{E}(L)$ to $\tau\sigma(B')$ so we have proved case i.
Case ii: There are no non-zero nilpotent elements in $B \cap B'$.
Then, 4.2.c and 16.3.a implies that $B \cap B' = T =$ toral (semisimple).
B is standard: $B(\Delta) = H + N$, $N(\Delta) = N$.
$[B, B] = N, T \cap N = 0$.
Thus, $N_B(T) = C_B(T)$.
Let C be a a CSA of $C_B(T)$. By (one of the) definitions of CSA we know it is self normalizing and N-nilpotent.
Thus, $T \subset N_{C_B(T)}(C) = C$.
In $n \in N_B(C')(\operatorname{ad} t)^k n = 0$. $t \subset T \subset C$.
$\operatorname{ad} t$-semisimple $\implies$ $k = 1, n \in C_B(T)$
$\operatorname{ad} t \cdot n = [t, n]$.
$\implies$ $N_B(C) = N_{C_B(T)}(C) = C$.
$\implies$ C is self-normalizing not only in the centralizer, but also in B. It is also nilpotent.
Therefore, C is a CSA of B.
C-maximal toral of L is conjugate under $\mathscr{E}(B)$ $\implies$ under $\mathscr{E}(L)$.
Thus, WLOG we can assume that $T \subset H$.
Suppose $T = H$. Then, $B' \supsetneq H$.
$\implies$ B' includes at least on e L_α with $\alpha < 0$ relative to Δ.
$\tau_\alpha(B') = B'', B'' \cap B \supset H + L_\alpha$
$\implies$ second induction hypothesis
B'' is conjugate to B under $\mathscr{E}(L)$
Let $T \subsetneq H$.
$B' \subset C_L(T)$
By first induction hypothesis we know it will have less degree.
$\dim C_L(T) < \dim L$.
$H \subset C_L(T)$ we can find a boorel subalgebra B'' of $C_L(T)$ that will contain H.
$\implies$ B' and B'' are conjugate under $\mathscr{E}(L, C_L(T)) \subset \mathscr{E}(L)$.
$B' \subsetneq C_L(T), T = B \cap B'$.
We can find an eigenvector $x \in B'$ for $\operatorname{ad} T$ and $t \in T$ such that $[t, x] = ax$. $a \in \mathbb{Q}_+$.
$S := H + \sqcup_{\alpha \in \Phi} L_\alpha$
$\alpha(t) \in \mathbb{Q}_+$.
S is subalgebra of L.
Similarly to llemma 16.3.c.2, S is solvable.
$B'' \supset C$ - Borel subalgebra.
$B'' \cap B' \supset T + Fx \supsetneq T = B \cap B'$ here x is eigenvector
$\dim B'' \cap B' > \dim B \cap B'$
Second induction hypothesis $\implies$ B'' is conjugate to B.
Similarly, we can prove that B'' is conjugate to B'
Thus, B is conjugate to B'.
(2) $B \cap B' = 0$

Then $\dim L \geq \dim B + \dim B'$.
B is standard so $\dim B \geq \frac{1}{2} \dim L$.
Let T-maximal toral subalgebra of B'.
Assume $T = 0 \implies B$ only has nilpotent elements $\implies$ by Engel's theorem, B has to be nilpotent.
By lemma 16.3.A, since B is borel, it is self normalizing $B = N_L(B)$.
Then B is a CSA of L.
15.3 $\implies$ all of CSAs of L are toral.
Being toral and nilpotent is a contradiction.
Thus, $T \neq 0$.
$T \subset H_0$-maximal toral of L
$\implies B \cap B'' \neq 0 \implies B'$ is conjugate to B.
$\dim B' = \dim B'' > \frac{1}{2} \dim L \implies$ we have a contradiction. □

There's a relationship between $\mathscr{E}(L)$ and the inner automorphisms.
Suppose L is semisimple lie algebra, H-CSA of L, Δ-base, Φ-root system
Let $\tau \in \operatorname{Aut} L$. We can see that $\tau(B)$ is conjugate to B by $\sigma_1 \in \mathscr{E}(L)$.
We can find $\sigma_2 \in \mathscr{E}(L, B) \subset \mathscr{E}(L)$ that sends $\sigma_1\tau(H)$ to H by 16.2.
$\sigma_2\sigma_1\tau$ preserves H and $B \implies$ it induces an automorphism on Φ. Leave Δ invariant.
Let ρ be such automorphism. It is not unique, but $\rho\sigma_2\sigma_1\tau(x_\alpha) = c_\alpha x_\alpha$ $[\alpha > 0]$, $\rho\sigma_2\sigma_1\tau(y_\alpha) = c_\alpha^{-1} y_\alpha$
$\rho\sigma_2\sigma_1\tau(h_\alpha) = h_\alpha$
τ differes from $\mathscr{E}(L) \cdot \Gamma(L)$ by a diagonal automorphism.
And diagonal automorphisms are inner automorphisms.
Therefore, $\operatorname{Aut}(L) = \operatorname{Inn}(L).\Gamma(L)$.
Note: in the semidirect case, $\mathscr{E}(L) = \operatorname{Inn}(L)$.

12 Thursday, 2/6/2025, Universal Enveloping Algebra by Hechi

Notation:
$\mathbb{F}$ = field, $\mathcal{L}/\mathbb{F}$ lie algebra, $V/\mathbb{F}$ vector space.

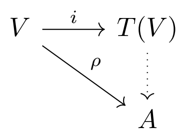
Definition (Tensor Algebra)**.** $T^m V = V^{\otimes m}$.

$$T^0 V = \mathbb{F}, T^1 V = V,$$

$$T^m V = \underbrace{V \otimes \cdots \otimes V}_{m \text{ copies}}$$

$T(V) = \coprod_{i=0}^{\infty} T^i V$
Multiplication by Tensor Product.

Universal Property:



A is an $\mathbb{F}$-algebra with 1.
Symmetric Algebra:
Let $I \subset T(V)$ be the two-sided ideal generated by all elements of the form:

$$x \otimes y - y \otimes x, x, y \in V$$

We define $S(V) = T(V)/I$.
We can write:

$$S(V) = \coprod_{i=0}^{\infty} S^i V$$

By writing it as a direct sum of terms of a degree.
$S^0V = \mathbb{F}, S^1V = V$ [since I contains deg ≥ 2 terms, they remain unchanged].
We can wrie:

$$I = \coprod_{i=2}^{\infty} I^i, I^i = I \cap T^i$$

This also enjoys a Universal Property: when A is commutative:

$$\begin{array}{ccc} T & \xrightarrow{i} & T(V) \\ & \searrow^{\rho} & \vdots \\ & & A \end{array}$$

If V is a finite dimensional vector space then $S(V)$ are polynomials!

Definition (Universal Enveloping Algebra)**.** An universal enveloping algebra is the pair (U, i) where U is an algebra and $i : \mathcal{L} \to U$ such that:

$$i([xy]) = i(x)i(y) - i(y)i(x) \qquad *$$

We also must have a universal property: suppose A is an $\mathbb{F}$-algebra with 1. Then, if $j : \mathcal{L} \to A$ satisfies $(*)$ then,

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{i} & U \\ & \searrow^{j} & \downarrow \phi \\ & & A \end{array}$$

From this we also know that (U, i) is unique upto isomorphism.

Construction of the Universal Enveloping Algebra

Let $J \subset T(\mathcal{L})$ be the 2-sided ideal generated by elements $x \otimes y - y \otimes x - [xy]$.
Define $U(\mathcal{L}) \coloneqq T(\mathcal{L})/J$. Then $U(L)$ satisfies $(*)$.
But this is not very explicit. We can explicitly construct it using the PBW theorem.
From now on let $T \coloneqq T(\mathcal{L}), S \coloneqq S(\mathcal{L}), U = U(\mathcal{L})$. We have a canonical projection $\pi : T \to U$.
Let $T^m = T^m\mathcal{L}, S^m = S^m\mathcal{L}$.
We also define the following filtrations:

$$T_m = T^0 \oplus \cdots \oplus T^m$$

$$U_m = \pi(T_m) \subset U, U_{-1} = 0$$

Facts: $U_m U_p \subset U_{m+p}$. $U_m \subset U_{m+1}$.
Thus it makes sense to define $G^m = U_m/U_{m-1}$. This is a $\mathbb{F}$-vector space.
The multiplication on U induces a well defined map:

$$G^m \times G^p \to G^{m+p}$$

Since lower degree terms just become 0.
We can extend this to $G = \coprod_{i=0}^{\infty} G^i$
Then, we have multiplication on G:

$$G \otimes G \to G$$

This gives G an $\mathbb{F}$-algebra structure. The algebra is abelian.

We can define the map:

$$\phi_m : T^m \to U_m \to G^m$$

Combining all the m, we have a surjective homomorphism:

$$\phi : T \twoheadrightarrow G$$

Lemma 11 (17.3)**.** $\phi(I) = 0$ and therefore ϕ induces a surjective map $\omega : S \twoheadrightarrow G$.

Proof. By definition of J,

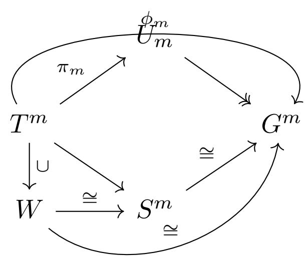
$$\pi(\underbrace{x \otimes y - y \otimes x}_{\text{deg } 2}) = \pi(\underbrace{[xy]}_{\text{deg } 1})$$

Thus, $\phi(x \otimes y - y \otimes x) \in U_1/U_1 \subseteq U_2/U_1$ thus $\phi(x \otimes y - y \otimes x) = 0$.
Hence the result. □

Theorem 12 (Poincaré-Birkhoff-Witt, 17.3)**.** $\omega : S \xrightarrow{\cong} G$.

Corollary 13 (17.3A)**.** We want to give a basis of U. If $W \subset T^m$ is a subspace such that $T^m \to S^m$ sends $W \xrightarrow{\cong} S^m$ then $\pi(W)$ is a complement of U_{m-1} in U_m

Proof. The following diagram commutes by construction



Bottom map gives $W \xrightarrow{\cong} G^m$. Top map, then $\phi_m(W)$ must be complement of U_{m-1} □

Corollary 14 (17.3B)**.** $i : \mathcal{L} \to U$ is injective.

Proof. Take $W = T^1$. □

Corollary 15 (17.3C)**.** This is traditionally known as the PBW theorem. Assume $\mathcal{L}$ has a countable basis $(x_1, \cdots)$. Then,
$\{x_{\sigma(1)} \cdots x_{\sigma(m)}\}, m \in \mathbb{Z}_{\geq 0}, \sigma$ permutation so that $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(m)$ is a basis for U.

Proof. Let $W = \text{span}\left\{x_{\xi(1)} \otimes \cdots \otimes x_{\sigma(m)} : \sigma(1) \leq \cdots \leq \sigma(m)\right\} \subseteq T^m$.
Clearly $W \xrightarrow{\cong} S^m$. Then we use corollary 17.3A. □

Proof of PBW. By well-ordering-principle we can define $(\chi_\lambda, \lambda \in \Omega)$ be an ordered basis of $\mathcal{L}$. This gives isomorphism $S \cong \mathbb{F}[z_\lambda]_{\lambda \in \Omega}$ where z_λ are just variables indexed by Ω.
Let $\Sigma = (\lambda_1, \cdots, \lambda_m)$ index with length m. Then $z_\Sigma = z_{\lambda_1} \cdots z_{\lambda_m} \in S^m$.
$x_\sigma = x_{\lambda_1} \otimes \cdots \otimes x_{\lambda_m} \in T^m$.
We say Σ is increasing if $\lambda_1 \leq \cdots \leq \lambda_m$ or $\varnothing$.
We define $z_\varnothing = 1$.
$\{z_\Sigma : \Sigma \text{ is increasing}\}$ is a basis of S.
We say $\lambda \leq \Sigma$ if $\lambda \leq \mu \forall \mu \in \Sigma$.
The idea is to give S a structure of L-module.

Lemma 16 (17.4A)**.** Fix $m \in \mathbb{Z}_{>0}$. Then there exists a unique linear map f_m : $\mathcal{L} \otimes S_m \to S$ with the following properties:
A_m: $f_m(\chi_\lambda \otimes z_\Sigma) = z_\lambda z_\Sigma \forall \lambda \leq \Sigma, z_\sigma \in S^m$.
B_m: $f_m(\chi_\lambda \otimes z_\Sigma) - z_\lambda z_\Sigma \in S_k \forall k \leq m, z_\Sigma \in S_k$
C_m: $f_m(\chi_\lambda \otimes f_m(x_\mu \otimes z_\tau)) = f_m(x_\mu \otimes f_m(\chi_\lambda \otimes z_\tau)) + f_m([\chi_\lambda x_\mu] \otimes z_\tau) \forall z_\tau \in S_{m-1}$.

Proof of Lemma: induction.

Lemma 17 (17.4B)**.** $\exists$ representation $\rho : \mathcal{L} \to \mathfrak{gl}(S)$ with:

a) $\rho(\chi_\lambda) z_\Sigma = z_\lambda z_\sigma, \forall k \lambda \leq \Sigma$

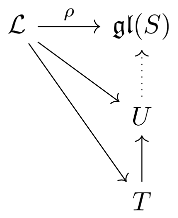
b) similar

c) similar

Proof of Lemma: Comining A_m, B_m, C_m for all m.

Lemma 18 (17.4C)**.** Let $t \in T_m \cap J$, then t_m (homogeneous degree m part of t) is in I.

Proof of Lemma: Can be seen from the commutative diagram:

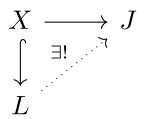


$J \subset \ker(T \to \mathfrak{gl}(S))$ so $\rho(t) = 0$. Then we can write t_m as linear combination of x_Σ, Σ has length m.
Then look at the highest deggree term of $\rho(t)$. It is a linear combination of z_Σ. But this term is 0. So $t_m \in I$.
We're done with all the lemma. We finally prove PBW.
Apply 17.4C to $t - t'$.
Let $t \in T^m$ such that $\pi(t) \in U_{m-1}$.
Then $\exists t'$ soo that $\pi(t) = \pi(t')$. Note that t' has degree strictly smaller than t.
Then, $t - t' \in J$.
Thus it satisfies the condition of 17.4C.
Therefore the m-degree part of $t - t'$ must be t [since $t \in T^m$].
Thus, by C, $t \in I$.

□

13 Thursday, 2/13/2025, Universal Enveloping Algebra by Hechi

Definition. Given a set X a lie algebra J is free on X if:



Construction: Let V be the vector space with a basis labeled by X, so $V = \mathbb{F}^{(X)}$.
Let $T(V)$ be the tensor algebra. Then $\mathcal{L}$ is the lie subalgebra generated by X.

Serre's Theorem

Let $\mathcal{L}$ be a lie algebra over an algebraically closed characteristic 0 field. $\mathcal{L}, H, \Delta = \{\alpha_1, \cdots, \alpha_l\}$. $x_i \in \mathcal{L}_{\alpha_i}, y_i \in \mathcal{L}_{\alpha_i}, h_i \in H$ where,
S_1: $[h_i h_j] = 0$
S_2: $[x_i y_i] = h_i, [x_i y_j] = 0, i \neq j$
S_3: $[h_i x_j] = \langle ; \alpha_j . \alpha_i \rangle x_j, [h_i y_j] = -\langle \alpha_j, \alpha_i \rangle u_j$
S_{ij}^+, S_{ij}^-

Theorem 19 (Serre's Theorem)**.** Given root system Φ with basis Δ, there exists $\mathcal{L}$ generated by $\{x_i, y_i, h_i\}$ satisfying $S_1, S_2, S_3, S_{ij}^{\pm}$ and $\mathcal{L}$ has root system Φ. ($\mathcal{L}$ has CSA H etc).

Idea: use the free Lie algebra and quotient by relations.
By the classification of root systems, there are only:
$A_l (l \geq 1)$
$B_l (l \geq 2)$
$C_l (l \geq 3)$
$D_l (l \geq 4)$
E_6
E_7
E_8
F_4
G_2
The first four are given in the first chapter.
Only thing needed to show is that they are semisimple!
From now we assume $\dim L < \infty$.

Theorem 20. L is called reductive if $\mathrm{rad}(L) = Z(L)$. If L is reductive then,

$$L = [LL] \oplus Z(L)$$

Proof. Note that $[L/Z(L), L/Z(L)] = L/Z(L)$ since $Z(L) = \mathrm{rad}(L)$. Thus, $[LL]$ maps surjectively to $L/Z(L)$ by the canonical projection.
We also know that $L' = L/Z(L)$ acts on L by the adjoint action. Thus, $L = M \oplus Z(L)$ for some M. We want to show that $M = [LL]$.
$[LL] = [M \oplus Z(L), M \oplus Z(L)] = [MM] \subseteq M$. But $[LL]$ maps surjectively to $L/Z(L)$, which isn't possible if it is properly contained.
Therefore, $M = [LL]$. Thus we have the isomorphism. □

Theorem 21. If $L \subseteq \mathfrak{gl}(V)$ and L acts irreducibly on V, then L is reductive and $\dim Z(L) \leq 1$.

Corollary 22.

If moreover $L \subseteq \mathfrak{sl}(V)$ then L is semisimple.

Definition. Suppose L is semisimple. L is called simply-laced if it's Dynkin diagram has only simple edges.
This is equivalent to, if $\alpha, \beta \in \Delta : \langle \alpha . \beta^\wedge \rangle = 2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \{0, -1, -2\}$.
Then, the simply laced are: A_l, D_l, E_6, E_7, E_8.
The letters here are A, D, E. This gives us the 'A-D-E' classification which is very common in algebra.

14 Thursday, 2/27/2025, Representation Theory by Zoia

Notation:
L-Semisimple Lie Algebra over F-algebraically closed field of char 0.
H-fixed CSA of L
H^*-dual space of the CSA H.

Φ: the root system, $\Delta = \{\alpha_1, \cdots, \alpha_l\}$, base of Φ.
$\mathcal{W}$- the Weyl Group.
Weight Spaces:
V- finite dim L-module.
H acts diagonally on .
$V = \bigoplus_\lambda V_\lambda$ where λ runs over H^*.
$V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v \forall h \in H\}$.
If $V_\lambda \neq 0$ we call V_λ a weight space. λ is the weight of H on V.
V'- the sum of all weight spaces V_λ [always direct].
Examples:

1) Consider L as an L-submodule via adjoint representation.

   Weights are the roots of $\alpha \in \Phi$ with weight spaces L_α of dim 1.

   If $L = \mathfrak{sl}(2, F)$ then a linear functional λ on H is completely defined by $\lambda(h)$ at the basis vector h.

Exercise 1: If V is an arbitrary L-module, then the sum of its weight space is direct.

Lemma 23. Let V be an arbitrary L-module. Then,

a) L_α maps V_λ into $V_{\lambda+\alpha}$ with $\lambda \in H^*, \alpha \in \varphi$.

b) The sum $V' = \sum_{\lambda \in H^*} V_\lambda$ is diret, and V' is an L-submodule of V.

c) If $\dim V < \infty$ then $V = V'$ [ex1]

d) If $x \in L_\alpha, v \in V_\lambda, h \in H$,

$$h \cdot \times \cdot v = x \cdot h \cdot v + [hx] \cdot v = (\lambda(h) + \alpha(h))xv$$

   Thus, L_α sends V_λ to $V_{\lambda+\alpha}$.

e) $L_\alpha, \alpha \in \Phi$ permutes the weight spaes.

Standard Cyclic Modules

Definition. A maximal vector (of weight λ) in an L-module V is a nonzero $v^+ \in V_\lambda$ is killed by all $L_\alpha(\alpha \in \Phi^+), \alpha \in \Delta$.
Meaning, $x_\alpha v^+ = 0 \, \forall x_\alpha \in L_\alpha$

Example: If L is simple and β is the maximal root in Φ relative to Δ then any nonzero element from L_β is a <u>maximal vector</u> (for the adjoint representation of L).
Note that the following fact about the Borel Subalgebra

$$B(\Delta) = H + \bigoplus_{\alpha \succ 0} L_\alpha$$

has a common eigenvector: which is a maximal vector.

Definition. If $V = \mathcal{U}(L) \cdot v^+$ for v^+ maximal vector of weight λ then we say V is standard cyclic.
Then λ is the highest weight on V.

Example: If V is an arbitrary A-module, then the sum of its weight spaces is direct.
Structure of such a submodule:
Fix nonzero $x_\alpha \in L_\alpha, \alpha \succ 0$ and choose $y_\alpha \in L_\alpha$ uniquely so that $[x_\alpha y_\alpha] = h_\alpha$.
Recall we have a partial order:
$\lambda \succ \mu \iff \lambda - \mu$ is a sum of positive roots.

Theorem 24. Consider $V, v^+ \in V_\lambda, \Phi^+ = \{\beta_1, \cdots, \beta_m\}$. Then,

a) V is spanned by vectors $y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} \cdot v^+$ where $i_j \in \mathbb{Z}_{\geq 0}$.

b) The weights of of V are of the form:

$$\mu = \lambda - \sum_{i=1}^{l} k_i \alpha_i \; \lambda - \mu \succ 0 \forall \lambda$$

c) $\forall \mu \in H^*, \dim V_\mu < \infty$ and $\dim V_\lambda = 1$.

d) Each submodule of V is the direct sum of its weight spaces.

e) V is indecomposable with a unique maximal submodule and unique irreducible quotient.

f) Every nonzero homo-c image of V is also standard cyclic of weight λ.

Proof. a) $L = B(\Delta) + \bigoplus_{\alpha \prec 0} L_\alpha$. Then by the PBW theorem, it holds that $\mathcal{U}(L) \cdot v^+ = \mathcal{U}\left(\bigoplus_{\alpha \prec 0} L_\alpha\right) \mathcal{U}(B(\Delta)) \cdot v^+ = \mathcal{U}\left(\bigoplus_{\alpha \prec 0} L_\alpha\right) \cdot F v^+$ [recall v^+ is a common eigenvector for B].

b) Consider the vector $y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} \cdot v^+$ has weight $\lambda - \sum i_j \beta_j$ Rewrite β as a non-negative $\mathbb{Z}$-linear combination of simple roots.

c) $\exists$ only finite number of polynomials $\sum i_j \beta_j$ from part b. These span the weight space V_μ, if $\mu = \lambda - \sum_i k_i \alpha_i$. The only vector of the form $\sum_j i_j \beta_j$ which has weight $\mu = \lambda$ is v^+.

d) Let W be a submodule of V and $w \in W$ such that w is a sum of v_i such that $v_i \in V_{\mu_i}$ and all the weights μ_i are distinct. We want to show that all $v_i \in W$.

Let $w = v_1 + \cdots + v_n$. We apply contradiction. WLOG suppose there exists $v_2 \notin W$.

Suppose $h \in H$ such that $\mu_1(h) \neq \mu_2(h)$.

$h \cdot w = \sum_i \mu_i(h) v_i \in W \implies (h - \mu_1(h) \cdot L) \cdot w = (\mu_2(h) - \mu_1(h)) v_2 + \cdots + (\mu_n(h) - \mu_1(h)) \cdot v_n \neq 0$

Then $v_2 \in W$. Contradiction.

e) By c and d each proper submodule of V lies in the sum of lies in the sum of weight spaces other than V_λ. Then the sum W of all submodule is still proper.

Thus V has a unique maximal submodule and unique irreducible quotient.

Thus, V cannot be the direct sum of two proper submodules since each of them are contained in W.

□

15 Thursday, 3/13/2025, Representation Theory by Zoia

Existence and Uniqueness of Standard Cyclic Modules

WTS: $\forall \lambda \in H^*, \exists!$ irreducible cylic L-module of highest weight λ [which can be infinite dimensional].

Theorem 25. Let V, W be standard cyclic modules of highest weight λ. If V, W are irreducible, then they are isomorphic.

Proof. Consider their sum: the L-module X such that $X = V \oplus W$ such that v^+ and w^+ represent the maximal vectors if weight λ in V, W respectively.
Let $x^+ = (v^+, w^+) \in X$. Then, x^+ is the maximal vector of weight λ.
Let's consider Y the L-submodule of X generated by x^+. Then Y is standard cyclic as well.
Let's consider the projection maps $P : Y \to V, P' : Y \to W$.

In this case, P, P' are L-module homomorphisms.
Since $P(x^+) = v^+, P'(x^+) = w^+$ we can conclude that $\operatorname{im} P = V, \operatorname{im} P' = W$.
Since V, W are irreducible quotients of a standard cyclic module Y, V and W must be isomorphic by the previous theorem. □

Construction of $Z(\lambda)$ by generators and relations

Also sometimes called Verma Modules.
Note that construction directly proves existence!
We define:

$$Z(\lambda) = U(L) \otimes_{U(B)} D_\lambda$$

where D_λ is a one dimensional vector space having v^+ as basis, and define action of B on D_λ by $(h + \sum_{\alpha \succ 0} X_\alpha) \cdot v^+ = h \cdot v^+ = \lambda(h)v^+$ for fixed $\lambda \in H^*$.
Choose nonzero element $x_\alpha \in L_\alpha$ where $\alpha \prec 0$.
Let $I(\lambda)$ be the left ideal in $U(L)$ generated by all such x_α along with $h_\alpha - \lambda(h_\alpha) \cdot 1$ $(\alpha \in \Phi^+)$.
These generators annihilate v^+ of $Z(\lambda) \implies I(\lambda)$ also annihilates $Z(\lambda)$.
$\exists$ a canonical homomorphism of left $U(L)$-modules $U(L)/I(\lambda) \to Z(\lambda)$ sending the coset of 1 onto v^+.
Meaning, $\overline{W} = W + I(\lambda) \mapsto W \otimes 1$.
PBW basis of $U(L) \implies$ we can see that this map sends the cosets of $U * B()$ onto $Fv^+ \implies$ this is one-to-one.
Thus, $Z(\lambda) \cong U(L)/I(\lambda)$.

Theorem 26 (Existence)**.** Let $\lambda \in H^*$ then $\exists$ an irreducible standard cyclic module $V(\lambda)$ of weight λ.

Proof. $Z(\lambda)$ is standard cyclic of weight λ and has a unique maximal submodule $Y(\lambda)$, and by the theorem in previous recitation. Then $V(\lambda) \equiv Z(\lambda)/Y(\lambda)$ is an irreducible and standard cyclic module of weight λ. □

Finite Dimensional Moddules

Necessary Conditions for finite dimensiona:
Let V be a finite dimensional irreducible L-module. Thus, $V \cong V(\lambda)$.
$\forall \alpha_i$ set $s_i(s_{\alpha_i})$ be the corresponding $\mathfrak{sl}(2, F)$ in L. Then, $s_i = L_{\alpha_i} \oplus L_{-\alpha_i} \oplus [L_{\alpha_i}, L_{-\alpha_i}]$.
We have $x_i \in L_{\alpha_i}, y_i \in L_{-\alpha_i}$ and set $h_i = [x_i, y_i]$. Then $[h_i, x_i] = 2x_i, [h_i, y_i] = -2y_i$.
This is how we get the copy: this subalgebra is isomorphic to $\mathfrak{sl}(2, F)$.
Then, $\lambda(h_i)$ determines completely $H_i \subset s_i$. By the theorem from 7.2, $\lambda(h_i)$ is a non-negative integer.

Theorem 27. If V is a finite dimensional irreducible L-module of highest weight λ then $\lambda(h_i)$ is a non-negative integer $1 \leq i \leq l$.

16 Thursday, 3/27/2025, Representation Theory by Zoia

Sufficient Condition for Finite Dimension

Lemma 28. Fix standard generators $\{x_i, y_i\}$ of L. Then the identities hold in $\mathcal{U}(L)$ for $k \geq 0; i \leq i, j \leq l$.

a) $[x_j, y_i^{k+1}] = 0$ if $i \neq j$

b) $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$

c) $[x_i, y_j^{k+1}] = -(k+1)y_i^k(kj - h_i)$

Proof. a is by 10.1 $\alpha_j - \alpha_i$ not a root.
b, c induction. □

Theorem 29. If $\lambda \in H^*$ is dominant integral then the irreducible L-module $V = V(\lambda)$ is finite dimensional, and its set of weights $\Pi(\lambda)$ is permuted by W, the weyl group with $\dim V_\mu = \dim V_{\sigma\mu}$ for $\sigma \in W$.

Proof. Denote by $\varphi : L \to gl(V)$. Fix a maximal vector v^+ of V of weight λ and $m_i = \lambda(h_i), 1 \le i \le l$
1. WTS: $y_i^{m_i+1}.v^+ = 0$.
Denote $\omega = y_i^{m_i+1} \cdot v^+$.
$x_j \cdot \omega = 0$
$x_i y_i^{m+1} \cdot v^+ = y_i^{m_i+1} x_i v^+$
$(m_i + D y_i^{m_i}) \cdot (m_i v^+ - m_i v^+) = 0$.
$\implies x_i\omega = 0$. If $\omega \neq 0$ then ω is a max vector in V with weight of $\lambda - (m_i+1)\alpha_i \neq \lambda$
$\implies$ contradition in 20.2.
$\implies \omega = 0$
2. For $1 \leftarrow i \le l$ V contains a nonzero fin. dimensional S_i module. The subspace spanned by $v^+, y_i v^+, y_i^{e_i} v^+, y_i^{m_i} v^+$ is stable under y_i according to 1. It is also stable under h_i sine each generator belongs to a weighr space of V. Thus, it is stable under h_i since ech belongs to a weight space of V. By c it is stable under x_i.
3. V is the sum of finite dimensional S_i-submmodules. Denote by V' the sum of all other submodules of V.
by 2 V' is nonzero. On the other hand, let W be any finite dimensional S_i-module of V. The span of all subspaces $x_\alpha(\alpha \in \Phi)$ is finite dimensional and S_i is stable under L.
Since V is stable under L, $V' = V$ by irreducibility.
4. For $1 \le i \le l$, $\varphi(x_i)$ and $\varphi(y_i)$ are locally nilpotent endomorphim of V.
If $v \in V \implies \delta$ is in finite sum of fin dim S_i submodules, $\varphi(x_i)$ and $\varphi(y_i)$ are nilpotent.
5. $S_i = \exp\varphi(x_i), \exp\varphi(-y_i)\exp\varphi(x_i)$ is a well defined automorphism of V by 4.
6. If μ is any weight of V then $S_i(V_\mu) = V_{\sigma\mu}$ where σ_i is reflect by d_i. V_μ lies in a fin. dim S_i-submodule V' and $S_i\big|_{V'}$ is the same as automorphism τ from 7.2.
7. The set of weights $\Pi(\lambda)$ is stable under W and $\dim V_\mu = \dim V_{\sigma\mu}[\mu \in \Pi(\lambda), \sigma \in W]$.
Since W is generated by $\sigma_1, \cdots, \sigma_L$ this follows from 6.
8. $\Pi(\lambda)$ is finite $\implies$ the set of W-conjugates of all dominant integral linear functions $\mu \prec \lambda$ is finite and $\Pi(\lambda)$ is in this set.
9. $\dim V$ is finite. by 20.2c, $\dim V_\mu$ is finite $\forall \mu \in \Pi(\lambda)$.
□

17 Thursday, 4/3/2025

Skipped

18 Thursday, 4/10/2025 by Hyeonmin

<u>Convention</u>
$\mathfrak{G}$ a semisimple lie algebra over an algebraically closed field F of char 0, $F = \mathbb{C}$.
$\mathfrak{h}$ a CSA, $\dim \mathfrak{h} = \ell, |\Phi^+| = m, \Phi \not\ni 0$.
$\mathfrak{n} = \bigoplus_{\alpha>0} \mathfrak{G}_\alpha, \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}, \mathfrak{n}^- = \bigoplus_{\alpha<0} \mathfrak{G}_\alpha$.
$\Lambda_r = \mathbb{Z}\Phi$.
$Z(\mathfrak{G})$ = the center of $U(\mathfrak{G})$.

1.1: Axioms and Consequences

Definition. The BGG Category $\mathcal{O}$ is the full subcategory of $\mathrm{Mod}_{U(\mathfrak{G})}$ satisfying:

$\mathcal{O}1)$ $M \in \mathcal{O}$ is a f.g. $U(\mathfrak{G})$-module.

$\mathcal{O}2$) $M \in \mathcal{O}$ is $\mathfrak{h}$-s.s., i.e. $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where $M_\lambda = \{x \in M \mid \forall h \in \mathfrak{h}, h.x = \lambda(h)x\}$.

$\mathcal{O}3$) $\forall v \in M, U(\mathfrak{n})v$ is finite dimensional.

Proposition 30. $M \in \mathcal{O}$ satiisfies:
$\mathcal{O}4$: All weight space M_λ are finite dimensional.
$\mathcal{O}5$: $\Pi(M) \coloneqq \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$ is contained in the union of finitely many sets of the form $\lambda - \Gamma$ where $\lambda \in \mathfrak{h}^*, \Gamma$ is the semigroup in Λ_r generated by Φ^+.

Proof. $\mathcal{O}1, \mathcal{O}2 \implies M$ has a finite generating set consisting weight vectors [from axiom 2, we have a direct sum. So, a vector in M can be represented as a finite sum of weight vectors by the definition of direct sum].
Thus, we may assume that $M = U(\mathfrak{G})v$ [M is generated by exactly one weight vector] with a weight vector v of weight λ.
$\mathcal{O}3 \implies V = U(\mathfrak{n})v$ is finite dimensional.
$U(\mathfrak{h})$ is stable on this so $U(\mathfrak{b})v$ is finite dimensional.
Since the action of $U(\mathfrak{n}^-)$ produces weights lower, then $\exists \mu \in \mathfrak{h}^* : \Pi(M) \subseteq \mu - \Gamma$ [in fact $\mu - \lambda$]. This gives us $\mathcal{O}5$.
Write $\forall$ weight $v, M_v = \{u \cdot w \mid u \in U(\mathfrak{n}^-), w \in V, \text{wt}(u \cdot w) = v\}$
Since $\forall w \in V, \exists$ only finitely many monomials $u = y_1^{i_1} \cdots y_m^{i_m}$ such that $\text{wt}(u \cdot w) = v$ and $V =$ finite dimensional, then M_v is finite dimensional [$\mathcal{O}4$]. □

Theorem 31 (1.1)**.** Category $\mathcal{O}$ satisies:

a) $\mathcal{O}$ is a noetherian category.

b) $\mathcal{O}$ is closed under submodules, quotients, finite direct sums.

c) $\mathcal{O}$ is an abelian cateogry.

d) If L is finite dimensional $\mathfrak{G}$-module, then $\mathcal{O} \xrightarrow{L \otimes_{\mathbb{C}} -} \mathcal{O}$ is an exact functor.

e) $M \in \mathcal{O}$ is a $Z(\mathfrak{G})$-finite. i.e., $\forall v \in M, \text{span}(Z(\mathfrak{G})v)$ is finite dimensional.

f) $M \in \mathcal{O}$ is finitely generated $U(\mathfrak{n}^-)$-module.

Proof. a) $U(\mathfrak{G})$ is a noetherian ring. [It has a filtration which we can pass to a graded ring. Which is isomorphic to a polynomial ring.] We can invoke $\mathcal{O}1 \implies \mathcal{O}$ is a noetherian category.

b) Quotient and finite direct sum ok. $U(\mathfrak{G})$ nnoetherian so submodule f.g.

c) Since $\text{Mod}_{U(\mathfrak{G})}$ is an abelian cattegory, we only need to check $\exists$ kernels, cokernels, finite diret sums. (b) gives us this.

d) $\mathcal{O}2 : (L \otimes M)_\nu = \bigoplus_{\mu + \lambda = \nu} (L_\mu \otimes M_\lambda)v$. $\mathcal{O}3$ ok. $\mathcal{O}1 : \{v_1, \cdots, v_n\}$ basis of $L, \{w_1, \cdots, w_p\}$ generating set of M. Let $N \coloneqq$ the submodule of $L \otimes M$ generated by $\{v_i \otimes w_j\}$. WTS: $L \otimes M = N$. We already know $N \subseteq L \otimes M$. Now, for any $v \in L$, since L is just a finite dimensional vector space, $v \otimes w_j \in N$ for any j. $\forall x \in \mathfrak{G}, x \cdot (v \otimes w_j) = (x \cdot v) \otimes w_j + v \otimes (x \cdot w_j)$. Since $x \cdot (v \otimes w_j) \in N$ and $(x \cdot v) \otimes w_j \in N \implies v \otimes (x \cdot w_j) \in N$. Iterating, $\forall u \in U(\mathfrak{G}), v \otimes (u \cdot w_j) \in N$.

e) Since $v \in M$ is the direct sum of weight vectors, then may assume that $v \in M_\lambda$ for some $\lambda \in \mathfrak{h}^*$. Then $Z(\mathfrak{G}) \cdot v \in M_\lambda$. [$\forall z \in Z(\mathfrak{G}) : h \cdot (zv) = z(hv) = z(\lambda(h)v) = \lambda(h)(zv)$]. $\mathcal{O}4$ implies the result.

f) $\{m_1, \cdots, m_p\}$ a generating set of $M, N_0 = \text{span}\{m_1, \cdots, m_p\}, N = U(\mathfrak{b})N_0$. $\mathcal{O}3$ implies $U(\mathfrak{n})N_0$ f.g. Thus A basis of N generates M as a $U(\mathfrak{n}^-)$-module.

□

1.2: Highest Weight Modules

Definition. $M = U(\mathfrak{G}).v^+$ is a highest weight module where v^+ is maximal vector of λ.

Remark. $M \in \mathcal{O}, \exists$ a maximal vector v^+ in M.

Theorem 32 (1.2)**.** $M = U(\mathfrak{G}) \cdot v^+$ a highest weight module of weight λ.. Chose $y_i \neq 0$ in $G_{-\alpha_i}$. Then,

a) M is spanned by $y_1^{i_1} \cdots y_m^{i_m} v^+ (i_j \geq 0)$. Thus M is $\mathfrak{h}$-s.s.

b) All wt μ satisfies $\mu \leq \lambda$.

c) $\forall$ wt $\mu, \dim M_\mu < \infty, \dim M_\lambda = 1 \implies M \in \mathcal{O}$.

d) Each nonzero quotient is again a highest weight module of weight λ.

e) A submodule generated by a maximal vector $\mu < \lambda$ is proper.

f) M has a unique maximal submodule and unique simple quotient. M Mis indecomposable.

g) All simple highest weight module M of weight λ are isomorphic. Moreover, $\dim \operatorname{End}_{\mathcal{O}} M = 1$

Corollary 33 (1.2)**.** $M \neq \mathcal{O}$, nonzero. Then M has a filtration $0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ with nonzero quotients. Each of which is a highest weight module.

Sketch. $\forall V :=$ $\mathfrak{n}$-submodule generated by a finite generating set of M of weight vectors. $\mathcal{O}3$ implies finite dimensional. Now induct on $\dim V$. □

1.3 Verma modules and Simple modules

Definition. $\mathbb{C}_\lambda = \mathbb{C}$ a $\mathfrak{b}$-module:

- $\forall h \in \mathfrak{h}, \mathfrak{v} \in \mathbb{C}_\lambda : h \cdot v = \lambda(h) v$.
- $\forall n \in \mathfrak{n}, v \in C_\lambda : nv = 0$.

$M(\lambda) := U(\mathfrak{G}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$: a Verma Module.

- Finite $U(\mathfrak{n}^-)$-module of rank 1.
- $M(\lambda) = U(\mathfrak{G}).v^+$ where $v^+ = 1 \otimes 1$.

Remark. Let N be a finite dimensional $U(\mathfrak{b})$-module on which $\mathfrak{h}$ acts semisimply. Then $U(\mathfrak{G}) \otimes_{U(\mathfrak{b})} N \in \mathcal{O}$. This defines an exact functor: action of $\mathfrak{h} \to \mathcal{O}$.

Definition. $L(\lambda)$ [resp. $N(\lambda)$] is the unique simple quotient (resp. unique maximal submodule) of $M(\lambda)$ [from theorem 1.2(f)].

Theorem 34 (1.3)**.** Every simple module in $\mathcal{O}$ of maximal weight λ is isomorphic to $L(\lambda)$. Moreover, $\dim \operatorname{Hom}(L(\mu), L(\lambda)) = \delta_{\mu\lambda}$.

Proof. Let $M \in \mathcal{O}$, simple of maximal weight λ. Let $v^+ \in M_\lambda \implies U(\mathfrak{G})v^+ \in M$ highest weight module.
Theorem 1.2g implies the result.
When $\mu = \lambda$ we know that $\dim \operatorname{End}_{\mathcal{O}} M = 1$. by 1.2g.
When $\mu \neq \lambda$ we claim that $\operatorname{Hom}(L(\mu), L(\lambda)) = 0$.
Let $0 \neq f \in \operatorname{Hom}(L(\mu), L(\lambda))$. These are simple modules, thus $\ker f = 0, \operatorname{im} f = L(\lambda)$.
So this is in fact an isomorphism.
$\forall m \in L(\mu)_\nu, h \cdot f(m) = f(h \cdot m) = f(\nu(h) m) = \nu(h) f(m) \implies f(m) \in L(\lambda)_\nu$.
$f : L(\mu)_\nu \xrightarrow{\sim} L(\lambda)_\nu$.
So, μ is a weight of $L(\lambda)$. Therefore $\mu \leq \lambda$. Isomorphism implies $\lambda \leq \mu$. Thus $\mu = \lambda$.
Contradiction. □

19 Thursday, 4/17/2025

1.4 Maximal vectors in Verma modules

Proposition 35 (1.4)**.** Given $\lambda \in \mathfrak{h}^*$ and fixed $\alpha \in \Delta$, suppose $n =: \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 1}$. If v^+ is a maximal vector of weight λ in $M(\lambda)$ then $y_\alpha^{n+1} \cdot v^+$ is the maximal vector of weight $\mu = \lambda - (n+1)\alpha < \lambda$.
Thus, $\exists$ a nonzero hom $M(\mu) \to M(\lambda)$ whose image is in $N(\lambda)$.

Lemma 36 (1.4)**.** a) $\forall i \neq j, [x_j, y_i^{k+1}] = 0$.

b) $[h_j, y_i^{k+1}] = -(k+1)\alpha)i(h_j)y_i^{k+1}$

c) $[x_i, y_i^{k+1}] = -(k+1)y_i^k(k \cdot 1 - h_i)$.

Proof. maximality:

- $\alpha_i = \alpha : x_\alpha(y_\alpha^{n+1}v^+) = [x_\alpha, y_\alpha^{n+1}]v^+ + y_\alpha^{n+1}\cancel{x_\alpha v^+} \underset{c}{=} -(n+1)y_\alpha^n\cancel{(n - \lambda(h_\alpha))}v_+ = 0$.
- $\alpha_i \neq \alpha : x_i(y_\alpha^{n+1}v^+) = [x_i, y_\alpha^{n+1}]v^+ \underset{a}{=} 0$
- Weight of $y_\alpha^{n+1}v^+ = \mu$: $\forall 1 \leq j \leq l, h_j(y_\alpha^{n+1}v^+) = [h_j y_\alpha^{n+1}]v^+ + y_\alpha^{n+1}(\lambda(h_j)v^+) \overset{b}{=} \underbrace{(-(n+1)\alpha(h_j) + \lambda(h_j))}_{\mu(h_j)} y_\alpha^{n+1}v^+$

Consider $f : M(\mu) \to M(\lambda)$ by $v_\mu \mapsto y_\alpha^{n+1}v^+$.
Then $f(M(\mu)) = U(\mathfrak{g}) \cdot y_\alpha^{n+1} \cdot v^+ =$ a proper submodule $\subseteq N(\lambda)$ □

Corollary 37 (1.4)**.** Let v^+ be instead a maximal vector of weight λ in $L(\lambda)$. Then $y_\alpha^{n+1}v^+ = 0$.

Proof. $L(\lambda)$ is simple so there doesn't exist maximal vector of $\mu < \lambda$. □

1.5 $\mathfrak{sl}_2(\mathbb{C})$

Fix the standard basis $\{h, x, y\}$.
$\dim \mathfrak{h}^* = 1 \implies h^* \xrightarrow{\cong} \mathbb{C}, \lambda \mapsto \lambda(h)$.
Identically, $\Lambda = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \forall \alpha \in \Phi\}$ with $\mathbb{Z}$ and identically $\Lambda_r = \mathbb{Z}\Phi$ with $2\mathbb{Z}$.
eg $\Phi = \{\alpha, -\alpha\}; \alpha(h) = 2 \implies \rho = \frac{\alpha}{2} \implies \Lambda = \mathbb{Z}\rho, \Lambda_r = \mathbb{Z}\alpha = 2\mathbb{Z}\rho$.
$M(\lambda)$ has weights $\lambda, \lambda - 2, \lambda - 4$ each with mul 1.
Basis vector ($i \geq 0$) for $M(\lambda)$ can be chosen so that ($v_{-1} = 0$):

- $h \cdot v_i = (\lambda - 2i)v_i$
- $x \cdot v_i = (\lambda - i + 1)v_{i-1}$
- $y \cdot v_i = (i+1)v_{i+1}$

Claim 1: $\dim L(\lambda) < \infty$ iff $\lambda \in \mathbb{Z}_{\geq 0}$.
Note: weight of $L(\lambda) : \lambda, \lambda - 2, \cdots, -\lambda$
Therefore, $N(\lambda) \cong L(-\lambda - 2)$
Claim 2: $M(\lambda)$ simple iff $\lambda \notin \mathbb{Z}_{\geq 0}$.
$\implies$ is done. $\impliedby$: suppose $M(\lambda)$ is not simple. Then $\exists N \subsetneq M(\lambda)$ having a maximal vector w which is not in $\mathbb{C}v^+$.
Then, $\exists k \in \mathbb{Z}_{\geq 0} : w = y^{k+1}v^+$ (up to scalar). Then $0 = xw = [x, y^{k+1}]v^+ = -(k+1)y^k(k - \lambda)v^+$.
Thus $k = \lambda$.

1.6 Finite Dimensional Modules

Theorem 38 (1.6)**.** TFAE:

a) $L(\lambda)$ fin dim

b) $\lambda \in \Lambda^+ = \{\lambda \in \mathfrak{h}^* \mid \forall \alpha \in \Phi, \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$

c) $\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu} \forall w \in W, \mu \in \mathfrak{h}^*$.

1.7 Action of the Center

Definition. Let $M = M(\lambda)$ be gen by v^+. For $z \in Z(\mathfrak{g})$ define $\chi_\lambda(z) \in \mathbb{C} : zv^+ = \chi_\lambda(z)v^+$ [since $zv^+ \in M_\lambda, \dim M_\lambda = 1$].
Then $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ the central character associated with λ

Note: $\forall v \in M, zv = \chi_\lambda(z)v$ since $v = u \cdot v^+, u \in U(\mathfrak{n}^-)$ and $zu = uz$.
χ_λ: alg hom and $\ker \chi_\lambda$ is a maximal ideal in $Z(\mathfrak{g})$.
More generally, any alg hom $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is called a central character.

Definition. Let $pr : U(\mathfrak{g}) \to U(\mathfrak{h})$ be the projection by sending other monomials to 0.
Then $\xi = pr\big|_{Z(\mathfrak{g})}$ is called the Harish-Chandra homomorphism.

Note: $\forall z \in Z(\mathfrak{g}), \chi_\lambda(z) = \lambda(\xi(z))$.
Therefore $\chi_\lambda(z)v^+ = zv^+ = pr(z)v^+ = \lambda(pr(z))v^+$