Friday, 4/4/2025

Let $f \in k[X], X \subset \mathbb{A}^N$ an affine variety.
For $p \in X$, we had $d_p f \in \mathfrak{m}_p/\mathfrak{m}_p^2[\cong (T_pX)*]$ where,

$$d_p f = (f - f(p)) \pmod{\mathfrak{m}_p^2}$$

$d_p : k[x] \to (\mathfrak{m}_p/\mathfrak{m}_p^2)$ is a derivation!
Reason: $fg - f(p)g(p) = (f-f(p))g + f(p)(g-g(p)) = \underbrace{(f-f(p))(g-g(p))}_{\in \mathfrak{m}_p^2} + g(p)(f - f(p)) + f(p)(g-g(p))$
Thus, we can say:

$$d_p f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p) \cdot d_p x_j$$

Remark. Viewing $f : X \to \mathbb{A}^1$, we see that $d_p f = d_p f : T_pX \to T_{f(p)}\mathbb{A}^1 \cong k$.
Since $d_p f$ is the linear dual of map.
Note that,

$$\frac{\mathfrak{m}_{f(p)}}{\mathfrak{m}_{f(p)}^2} = k[(x - f(p))] \xrightarrow{f^*} \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}$$

$$x - f(p) \mapsto (y \mapsto f(y) - f(p))$$

$$[x - f(p)] \mapsto [f - f(p)] = d_p f$$

Hence, $\forall p \in X, d_p f \in \mathfrak{m}_p/\mathfrak{m}_p^2 \forall f \in k[x]$.
Hence, fixing $f \in k[x]$ and varying p,

$$p \mapsto d_p f \in \mathfrak{m}_p/\mathfrak{m}_p^2$$

Now let X be any variety.
We obtain a function $df : X \to \bigsqcup_{p\in X} \mathfrak{m}_p/\mathfrak{m}_p^2 = \bigsqcup_{p\in X}(T_pX)^*$.
Consider $\Phi(X) = \left\{\varphi : X \to \bigsqcup_{p\in X} T_p^*X \mid \varphi(p) \in T_p^*X \forall p \in X\right\}$

Lemma 1. $\Phi(X)$ is a $k[X]$-module.

Proof. Given $g \in k[X], \varphi \in \Phi(X)$, we define:

$$(g \cdot \varphi)(p) \coloneqq g(p)\varphi(p)$$

Check that this defines a $k[X]$-module structure. □

Definition. A regular 1-form $\omega \in \Phi(X)$ is an element such that $\forall p \in X, \exists p \in U \underset{\text{open, affine}}{\subset} X$ such that $\omega\big|_U$ is the $k[U]$-submodule of $\Phi(U)$ generated by $df, f \in k[U]$.

$$\Omega^1[X] = \{\text{reg. 1-forms on } X\} \underset{k[X]\text{-submodule}}{\subset} \Phi(X)$$

Examples: $\Omega^1[\mathbb{A}^n] = \bigoplus_{i=1}^n k[\mathbb{A}^n] \cdot dx_i$

Proof. $\forall p \in \mathbb{A}^n$, recall that $\{d_px_1, \cdots, d_px_n\}$ is a basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$.
Hence, any $\varphi \in \Phi(\mathbb{A}^n)$ can be written as $\sum_{i=1}^n \varphi_i \cdot dx_i$ where $\varphi_i \in \mathrm{Fun}(\mathbb{A}^n, k)$.
Then, if $\omega = \sum_{i=1}^n \omega_i dx_i \in \Omega^1[\mathbb{A}^n]$, then $\forall p \in \mathbb{A}^n, \exists U \ni p \underset{\text{open}}{\subset} \mathbb{A}^n$.
$\omega \in k[U].\{df \mid f \in k[U]\}$.
Since $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot dx_i$,
$f \in k[U] \implies df \in df \in \bigoplus_{i=1}^n k[U].dx_i$
$\implies \omega \in \bigoplus_{i=1}^n k[U].\,\mathrm{d}x_i \iff \omega_i \in k[U]$.
Since this holds $\forall p \in \mathbb{A}^n, \omega_i \in k[\mathbb{A}^n]$.

□

$X \in \mathbb{P}^1, \Omega^1[X] = 0$.

Proof. Note that $\mathbb{P}^1 = \underbrace{\mathbb{A}_0^1}_{(x_0=1)} \cup \underbrace{\mathbb{A}_1^1}_{(x_1=1)}$.

$\mathbb{P}^1 = \{(x_0 : x_1)\}$.
$t = \frac{x_1}{x_0}, u = \frac{x_0}{x_1}$.
By example 1, $\omega \in \Omega^1[X] \implies \omega\big|_{\mathbb{A}_0^1} \in \Omega^1[\mathbb{A}_0^1]$ so $\omega\big|_{\mathbb{A}_0^1} = p(t) \cdot dt$.
Similarly, $\omega\big|_{\mathbb{A}_1^1} = q(u) \cdot du$.
For these to define $\omega \in \Omega^1[X]$ we require $p(t)dt = q(u)du = q(1/t) - dt/t^2$ on $\Omega^1[\mathbb{A}_0^1 \cap \mathbb{A}_1^1]$ where $ut = 1$.
LHS is regular at 0.
RHS has a pole of order ≥ 2 at 0.
Thus this is only possible when $p = q = 0$.

□

Let $X \subset \mathbb{P}^2$ defined by $\xi_0^2 + \xi_1^2 + \xi_2^2 = 0$.
Since X is projective, $k[X] = k$. Nevertheless, $\Omega^1[X] \neq 0$!!

Proof. Let $\mathbb{A}_0^2 = \{\xi_0 \neq 0\}, \mathbb{A}_1^2 = \{\xi_1 \neq 0\}, \mathbb{A}_2^2 = \{\xi_2 \neq 0\}$
Let $X = U_{01} \cup U_{12} \cup U_{13}$ where $U_{ij} = \mathbb{A}_i^2 \cap \mathbb{A}_j^2$. Main idea: two of $\{\xi_0, \xi_1, \xi_2\}$ cannot be zero!
We define:
$U_{01} : x = \frac{\xi_1}{\xi_0}, y = \frac{\xi_2}{\xi_0}, \varphi = \frac{dy}{x^2}$.
$U_{12} : u = \frac{\xi_2}{\xi_1}, v = \frac{\xi_0}{\xi_1}, \psi = \frac{dv}{u^2}$.
$U_{02} : s = \frac{\xi_0}{\xi_2}, t = \frac{\xi_1}{\xi_2}, \chi = \frac{dt}{s^2}$
On $\mathbb{A}_0^2 \cap \mathbb{A}_1^2 \cap \mathbb{A}_2^2, = U_{01} \cap U_{12} = U_{01} \cap U_{02} = U_{12} \cap U_{02}, \varphi = \psi = \chi$.
Hence $\exists \omega \in \Omega^1[x]$ such that $\omega\big|_{U_{01}} = \varphi, \omega\big|_{U_{12}} = \psi, \omega\big|_{U_{02}} = \chi$. □

Theorem 2. Let $p \in X$ be nonsigunlar. Then $\exists U \ni p \underset{\text{affine open}}{\subset} X$ such that $\Omega^1[U]$ is a free $k[U]$-module of rank $n = \dim_p X$.

Proof. WLOG $X \underset{\text{affine}}{\subset} \mathbb{A}^n, X$ irreducible. Let $\mathfrak{a}_X = \{F_1, \cdots, F_m\}$.
$\dim T_p X = n = \dim X$.
Since $T_p X = \left\{(a_1, \cdots, a_N) \in \mathbb{A}^N \mid \sum_{j=1}^N \frac{\partial F_i}{\partial x_j}(p) a_j = 0\right\}$,

$$T_p X = \ker \left(\begin{bmatrix} \frac{\partial F_1}{\partial x_1}(p) & \cdots & \frac{\partial F_1}{\partial x_N}(p) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1}(p) & \cdots & \frac{\partial F_m}{\partial x_N}(p) \end{bmatrix} \right)$$

Hence, $\text{rank}(\cdots) = N - n$. □