

M503 Non-Commutative Algebra

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Office Hours: Wednesdays 4:30, RH316.

Tuesday, 1/13/2026

1 Introduction

We are interested in non-commutative algebras of finite dimension over a field.

Reference: Lorenz Algebra Vol. 2, chapters 28 and following.

General convention:

- i) Rings will always assumed to be associative and unital.
- ii) Modules are always left modules [unless stated otherwise]. Even when philosophically it's better to consider them as right modules we will turn them into the opposite ring.

R = commutative unital ring.

A = unital R -algebra, not necessarily commutative.

Recall: this means that A is an R -module, and one has $\forall x \in R, \forall a, b \in A, x \cdot (ab) = (xa) \cdot b = a \cdot (xb)$.

Remark 1.1. 1) $\varphi : R \rightarrow A, x \mapsto x \cdot 1_A$ is a ring homomorphism with image in $Z(A) = \{a \in A \mid \forall b \in A, ab = ba\}$.

$$(x \cdot 1) \cdot a = x \cdot (1 \cdot a) = x \cdot a = x \cdot (a \cdot a) = a \cdot (x \cdot 1_A)$$

- 2) Conversely, if $\psi : R \rightarrow B$ [where B is any unital ring] is an unital ring homomorphism with $\text{im}(\psi) \subset Z(B)$, then the multiplication $x \cdot b := \psi(x)b$ gives B the structure of an R -module and B becomes an R -algebra.
- 3) The map ψ in (1) need not be injective. $\mathbb{Z} = R, \mathbb{Z}/n\mathbb{Z} = A$ is allowed.
- 4) Subalgebras of an algebra contain the unit element by convention.

- 5) By an *ideal* in A we always mean a 2-sided ideal. We have left ideals and right ideals defined in the usual way.
- 6) A *division algebra* is an R -algebra A such that $\forall a \in A \setminus \{0_A\} \exists b \in A$ such that $ab = ba = 1_A$. It is often also called a skew field. The center of a division algebra is a field.
Modules over a division algebra will also be called vector spaces.
- 7) $A\text{-Mod}$ denotes the category of (left) A -modules.

Proposition 1.2. Let A be an R -algebra and $M, N \in A\text{-Mod}$. Set $H = \text{Hom}_A(M, N)$. Then H is an R -module with the following action: $R \times H \rightarrow H, (x, f) \mapsto [m \mapsto x \cdot f(m)]$. This gives H the structure of an R -module.

We need R to be commutative, because otherwise $am \mapsto xf(am) = xaf(m)$ which is not necessarily $axf(m)$.

If $N = M$ then $H = \text{End}_A(M)$ is in fact an R -algebra w.r.t. the same module structure and composition of maps as multiplication.

Proof. HW1 □

Remark 1.3. For $M \in A\text{-Mod}$ we can regard it also as a module over $\text{End}_R(M)$.

Notation 1.4. 1) $A_\ell = A$ considered as an A -module by left multiplication $A \times A_\ell \rightarrow A_\ell, (a, b) \mapsto ab$.

- 2) $A^o =$ opposite algebra of $A = A$, but with multiplication defined by $A \xrightarrow{A^o} A \cdot b := b \cdot a$. A^o is still an R -algebra.
We set $A_r = (A^o)_\ell$. This is just A but with A^o -module structure.

$$A^o \times A_r \rightarrow A_r, (a, b) \mapsto a \xrightarrow{A^o} b = ba$$

The R -algebra $M_n(A)$

$M_n(A)$ denotes the R -module of $n \times n$ -matrices with entries in A .

Multiplication: $(a_{ij})_{1 \leq i, j \leq n} \cdot (b_j)_{i \leq i, j \leq n} := (\sum_{k=1}^n a_{ik} b_{kj})_{1 \leq i, j \leq n}$.

Proposition 1.5. i) The map $\lambda : A \rightarrow \text{End}_R(A), \alpha(a) = [b \mapsto ab]$ is injective and induces an isomorphism of R -algebras $A \xrightarrow[\lambda]{\cong} \text{End}_{A^o}(A) \subset \text{End}_R(A)$.

ii) The map $\rho : A^o \rightarrow \text{End}_R(A), \rho(a) = [b \mapsto ba]$ is injective and induces an isomorphism of R -algebras $A^o \xrightarrow[\rho]{\cong} \text{End}_A(A) \subset \text{End}_R(A)$.

We will discuss this later.

Proposition 1.6. i) The map $\text{End}_A(A_\ell^{\oplus n}) \xrightarrow{\cong} M_n(A^o)$, given by $f \mapsto (a_{ij})_{i,j}$ where $f(e_j) = \sum_{i=1}^n a_{ij} e_i$ is an isomorphism of R -algebras.

ii) The map $\text{End}_{A^o}(A_r^{\oplus n}) \xrightarrow{\cong} M_n(A)$, given by $f \mapsto (a_{ij})_{i,j}$ where $f(e_j) = \sum_{i=1}^n a_{ij} e_i$ is an isomorphism of R -algebras.

Proof. i) Suppose $f \mapsto (a_{ij})_{i,j}, g \mapsto (a_{ij})_{i,j}$. Then $(f \circ g)(e_j) = f(g(e_j)) = f(\sum_{i=1}^n b_{ij} e_j) = \sum_{i=1}^n b_{ij} f(e_j) = \sum_{i=1}^n b_{ij} \sum_{k=1}^n a_{ki} e_k = \sum_{k=1}^n (\sum_{i=1}^n b_{ij} a_{ki}) e_k = \sum_{k=1}^n (\sum_{i=1}^n a_{ki} \xrightarrow{A^o} b_{ij}) e_k$. Note that $\sum_{i=1}^n a_{ki} \xrightarrow{A^o} b_{ij}$ is simply the entry in position (k, j) of $(a_{ij}) \cdot (b_{ij})$ where they're elements of $M_n(A^o)$.

Hence, $f \circ g \mapsto (a_{ij}) \xrightarrow{M_n(A^o)} (b_{ij})$.

ii) Is identical.

□

Now we generalize. Let $N \in A\text{-Mod}$ and consider N^n [which is a shorthand for $N^{\oplus n}$] as an $A\text{-Mod}$ by diagonal multiplication of A .

Let $\iota_j : N \rightarrow N^n$ be the inclusion of the j 'th summand, $x \mapsto (0, \dots, 0, x, 0, \dots, 0)$ where x is in the j 'th position.

Let $\pi_i : N^n \rightarrow N$ be the projection onto i 'th direct summand, $(x_1, \dots, x_n) \mapsto x_i$.

Proposition 1.7. The map $\text{End}_A(N^n) \xrightarrow{\cong} M_n(\text{End}_A(N))$ where $f \mapsto (f_{ij})_{1 \leq i, j \leq n}$, $f_{ij} = \pi_i \circ f \circ \iota_j$ is an isomorphism of R -algebras. Concretely, $(x_1, \dots, x_n) \mapsto \left(\sum_{j=1}^n f_{ij}(x_j) \right)_{i=1, \dots, n}$.

Moreover setting $C = \text{End}_A(N)$, $N' = N^n$, $C' = \text{End}_A(N')$ then the inclusion $\text{End}_C(N) = \text{End}_{\text{End}_A(N)}(N) \xrightarrow{\Delta} \text{End}_C(N') = \text{End}_{\text{End}_A(N)}(N^n) \cong M_n(\text{End}_C(N))$.

Here $\Delta : g \mapsto [(x_1, \dots, x_n) \mapsto (g(x_1), \dots, g(x_n))]$.

This induces an isomorphism (of R -algebras) $\text{End}_C(N) \rightarrow \text{End}_{C''C'}(N') = \text{End}_{\text{End}_A(N^n)}(N^n)$.

Note that $\text{End}_A(A_\ell) \cong A^o$, so this is sort of a generalization to the previous proposition.

Proof. The first assertion is an easy exercise.

Let $g \in \text{End}_C(N)$ and let $\tilde{g} : N^n \rightarrow N^n$ be $(x_1, \dots, x_n) \mapsto (g(x_1), \dots, g(x_n))$. We want to show that \tilde{g} commutes with elements of C' .

Let $f \in C' = \text{End}_A(N')$.

$$\begin{aligned} \tilde{g}(f(x_1, \dots, x_n)) &= \tilde{g}\left(\left(\sum_{j=1}^n f_{ij}(x_j)\right)\right)_i = \left(g\left(\sum_{j=1}^n f_{ij}(x_j)\right)\right)_i = \left(\sum_j g(f_{ij}(x_j))\right)_i \\ &= f((g(x_1), \dots, g(x_n))). \end{aligned}$$

Here $f_{ij} \in C$.

Surjectivity of $\text{End}_C(N) \rightarrow \text{End}_{C'}(N')$: given $h \in \text{End}_{C'}(N')[\subset \text{End}_C(N') \cong M_n(\text{End}_C(N))]$, we need to show that it is diagonal. Which means we have to show that $\forall i \neq j : \pi_i \circ h \circ \iota_j = 0$ and $\pi_i \circ h \circ \iota_i = \pi_j \circ h \circ \pi_j$.

$h \in \text{End}_{C'}(N^n) \implies h \text{ commutes with } \iota_j \circ \pi_i : N^n \rightarrow N^n$. Note that $\iota_j \circ \pi_i \in C' = \text{End}_A(N^n)$.

Then $h \circ \iota_j = h \circ \iota_j \circ \pi_i \circ \iota_i$.

Compose with π_i on the left: $\pi_i \circ h \circ \iota_j = \underbrace{\pi_i \circ \iota_j}_{=0} \omega \pi_i \circ h \circ \iota_i = 0$.

Compose with π_j on the left: $\pi_j \circ h \circ \iota_j = \pi_i \circ h \circ \iota_i$.

□

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Corollary 1.8. i) $\text{End}_A(A^n) \xrightarrow{\cong} M_n(A^o)$.

ii) $A \xrightarrow{\cong} \text{End}_{\text{End}_A(A^n)}(A^n)$.

iii) $A^o \cong \text{End}_{M_n(A)}(A^n)$.

iv) $Z(\text{End}_A(A^n)) = Z(A^o) \cdot \text{id}_{A^n} = Z(A) \cdot \text{id}_{A^n}$.

v) $Z(M_n(A)) = Z(A) \cdot 1_n$ where $1_n = n \times n$ identity matrix.

Proof. i) Follows from 1.7 and 1.5.

ii) Take $N = A_\ell$ in 1.7 and $C = \text{End}_A(A) \xrightarrow{1.5} A^o$.

$$\text{End}_C(N) = \text{End}_{A^o}(A_\ell) = A \xrightarrow[1.7]{\cong} \text{End}_{\text{End}_A(A_\ell^n)}(A_\ell^n).$$

iii to v are exercises. \square

Definition (Anti-isomorphism). α is an *anti-isomorphism* if it is an isomorphism of R -modules sending 1_A to 1_A and $\forall a, b \in A: \alpha(ab) = \alpha(b)\alpha(a)$.

Proposition 1.9. i) Suppose \exists anti-isomorphism $\alpha: A \rightarrow A$. Then α is an isomorphism $A \rightarrow A^o$.

ii) The map $M_n(A) \rightarrow M_n(A^o)^o$ given by $x \mapsto x^T$ is an isomorphism of algebras.

iii) If $A \cong A^o$ [as R -algebras], then $M_n(A) \cong M_n(A^o)$. In particular, matrix algebras over fields are isomorphic to their opposite.

Proof. HW1. \square

Example. 1) $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ the Hamilton Quaternions are isomorphic to \mathbb{H}^o has an anti-isomorphism given by $a + bi + cj + dk \mapsto a - bi - cj - dk$.

2) Let R be a field and $A = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in R \right\}$. Then $A^o \not\cong A$.

Definition. 1) A module N is called *simple* (or *irreducible*) if $N \neq 0$ and the only submodules of N are 0 and N .

2) A module N is called *semisimple* if it is isomorphic to a direct sum of (not necessarily finitely many) simple modules. Formally, we allow the empty direct sum. As a consequence, 0 is a semisimple module.

3) A submodule $N \subset M$ is called *minimal* (resp. *maximal*) if it is minimal (resp. maximal) among all non-zero submodules (resp. among all proper submodules). 0 doesn't have any minimal or maximal submodule.

Example. Let $R = K$ be a field. The simple K -modules are just the 1-dimensional vector spaces, and they're all isomorphic to K itself.

Recall that every vector space has a basis [assuming axiom of choice]. Therefore, every K -module is semisimple.

Same is true for division algebras.

Proposition 1.11. Let M be a *finitely generated* (f.g.) A -module, and $N_0 \subsetneq M$ a proper submodule. Then M has a maximal submodule N containing N_0 .

Proof. We use Zorn's Lemma.

Let $\mathcal{M} = \{N \subsetneq M : N \text{ submodule, } N \supseteq N_0\}$.

$N_0 \in \mathcal{M} \implies \mathcal{M} \neq \emptyset$.

Let $N_1 \subset N_2 \subset \dots \subset$ an ascending chain in \mathcal{M} . Define $\tilde{N} = \bigcup_{i \geq 1} N_i$.

Let m_1, m_2, \dots, m_r be generators of M . If $\tilde{N} = M$ then there exists $i \geq 1$ such that $m_1, \dots, m_r \in N_i$. But that would imply $N_i = M$. This is a contradiction.

Now we apply Zorn's Lemma. \mathcal{M} contains a maximal element N . This N is maximal among all proper submodules. Therefore, $N \supset N_0$ is maximal. \square

Proposition 1.12. Let $0 \neq N \in A\text{-Mod}$. TFAE:

- i) N is simple.
- ii) $\forall x \in N \setminus \{0\} : Ax = N$.
- iii) \exists maximal (left) ideal $I \subset A : N \underset{A}{\cong} A/I$.

Proof. Consider the map $A \rightarrow N$ given by $a \mapsto ax$. Let $I = \text{Ann}_A(x) \xrightarrow{N \text{ simple}} I$ is maximal. \square

Proposition 1.13. Let $n > 0$ and V be an n -dimensional vector space over a division algebra D , and set $A = \text{End}_D(V)$. Regard V as an A -module. Then,

- i) V is simple as an A -module.
- ii) $A_\ell \cong V^n$. In particular A_ℓ is a semisimple A -module.

Proof. i) We use the fact that if v_1, \dots, v_n is a basis of V over D , then the map $\text{End}_D(V) \rightarrow \underbrace{V \oplus \dots \oplus V}_n = V^n$ given by $a \mapsto (a(v_1), \dots, a(v_n))$ is a bijection of abelian groups.

It is not necessarily a bijection of D -vector spaces.

Given any $v_1 \in V \setminus \{0\}$ and any $w \in V$, we can extend $\{v_1\}$ to be a basis of V and find $a \in A$ such that $a(v_1) = w$. Then $Av_1 = V \xrightarrow{1.12} V$ is simple.

- ii) Note that the map $A = \text{End}_D(V) \rightarrow V^n$ in (i) is A -linear, hence an isomorphism of A -modules.

\square

Remark. Take in 1.13 $V = D^n$. Then $\text{End}_D(D^n) \underset{1.8}{\cong} M_n(D^o)$.

Then, 1.13 says $M_n(D^o) \cong \underbrace{D^n \oplus \dots \oplus D^n}_n$ where the D^n are columns of the matrices in the LHS.

Definition. 1) An algebra A is called *semisimple* if A_ℓ is a semisimple A -module.

- 2) A is called *simple*, if $A \neq 0$ and does not contain any (2-sided) ideals other than 0 and A .

Example. $M_n(D)$ where D is a division ring is a semisimple algebra by 1.13. This algebra is also simple.

Note that not every semisimple algebra is simple. Let A and B be semisimple. Exercise: $A \times B$ with componentwise addition and multiplication is semisimple.

Let K_1, \dots, K_n be fields (or skew fields). Then $K_1 \times \dots \times K_n$ is semisimple.

Caution: There are simple algebras which are not semisimple (HW1).

Proposition 1.14 (Schur's Lemma). Let $M, N \in A\text{-Mod}$. Set $H = \text{Hom}_A(M, N)$.

- i) If M is simple, any non-zero $f \in H$ is injective. If N is simple, any non-zero $f \in H$ is surjective.
- ii) If M, N are simple, then $M \cong N$ or $H = 0$.
- iii) If M is simple, then $\text{End}_A(M)$ is a division algebra.

Proof. Straightforward. □

Lemma 1.15. Suppose $M = \sum_{i \in I} N_i$ is the sum of a family $(N_i)_{i \in I}$ of *simple* submodules.

Let $N \subset M$ be any submodule. Then $\exists J \subset I$ such that,

$$M = N \oplus \bigoplus_{j \in J} N_j$$

Proof. Let $\mathcal{S} = \left\{ J \subset I : \text{s.t. } M + \sum_{j \in J} N_j = N \oplus \bigoplus_{j \in J} N_j \right\}$. Note that $\emptyset \in \mathcal{S} \implies \mathcal{S} \neq \emptyset$.

If $J_1 \subset J_2 \subset \dots$ is a chain in \mathcal{S} and $\tilde{J} = \bigcup_{k=1}^{\infty} J_k$ then [exercise] $\tilde{J} \in \mathcal{S}$.

Zorn's lemma implies that \mathcal{S} has a maximal element J .

Check: J works. □

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Proposition 1.16. For $M \in A\text{-Mod}$. TFAE:

- i) M is the sum of simple submodules.
- ii) M is semisimple.
- iii) Every submodule is a direct summand of M .

Proof. i \implies ii: take $N = 0$ in 1.15.

ii \implies iii: Apply 1.15 with the submodule as N .

iii \implies i: Let $x \in M \setminus \{0\}$ and consider the ‘cyclic submodule’ $C = Ax \subset M$. Note that $x \in C \implies 0 \neq C$.

1.11 implies \exists maximal submodule $L \subsetneq C$. Since L is a maximal submodule, C/L must be simple.

By assumption, $\exists N \subset M$ such that $M = L \oplus N$. Since $L \subset C$ it follows that $C = L \oplus (C \cap N)$.

Therefore, $C/L \cong C \cap N \implies C \cap N$ is a simple module.

It follows that every non-zero submodule of M contains a simple submodule.

Set $M' = \sum_{N \subset M \text{ simple}} N$. Let $M'' \subset M$ be the summand, i.e. $M = M' \oplus M''$.

If $M'' \neq 0$, it contains a simple submodule $N_0 \subseteq M''$. But then $N_0 \subset M'$. This is a contradiction.

Thus, $M'' = 0$. Which means $M = M' = \text{sum of simple submodules}$. \square

Remark. Given any $M \in A\text{-Mod}$, the *socle* of M , $\text{soc}(M)$ is defined to be the largest semisimple submodule of M .

By 1.16, $\text{soc}(M) = \sum_{N \subset M \text{ simple}} N$.

Proposition 1.17. Suppose M is the sum of simple submodules $(N_i)_{i \in I}$. Then,

- i) Every submodule or quotient module is isomorphic to $\bigoplus_{j \in J} N_j$ for some $J \subset I$. Note that it need not be directly equal, just isomorphic. Consider $K \xrightarrow{\Delta} K \oplus K$.
- ii) Any simple submodule $N \subset M$ is isomorphic to one of the N_i .

Proof. ii follows from i.

For i: Let $M \xrightarrow[\pi]{} M''$ be a quotient of M . Then, $\overline{N_i} := \pi(N_i)$ is zero or simple.

Set $\overline{I} := \{i \in I \mid \overline{N_i} \neq 0\}$.

$\implies M'' = \sum_{i \in \overline{I}} \overline{N_i}$. Applying 1.15, $\exists \overline{J} \subset \overline{I}$ so that $M'' = \bigoplus_{j \in \overline{J}} \overline{N_j} \cong \bigoplus_{j \in \overline{J}} N_j$.

If $M' \subset M$ is a submodule, 1.16 $\implies \exists N \subset M : M = N \oplus M'$. Then $M' \cong M/N$, which is a quotient of M . \square

Corollary 1.18. i) Every submodule or a quotient module of a semisimple module is semisimple.

- ii) If A is semisimple, any simple A -module is isomorphic to a submodule of A_ℓ , and so is isomorphic to a minimal left ideal of A .

Proof. i) Directly follows from 1.17.

- ii) Suppose A is semisimple. Then by definition, $A_\ell = \bigoplus_{i \in I} N_i$ where N_i are simple. If N is a simple module, 1.12 implies $N \cong A_\ell/I$ where $I \subset A$ is a maximal submodule. 1.17 implies that $N \cong N_i$ for some i .

Note that each N_i is a left ideal. Since it is simple, it must be minimal among the non-zero submodules.

\square

Proposition 1.19. For an algebra A TFAE:

- i) A is semisimple.
- ii) Every A -module is semisimple.

Proof. ii \implies i is direct.

i \implies ii: Suppose $(m_i)_{i \in I}$ is a set of generators of M as an A -module. Consider $A_\ell^{(I)} := \bigoplus_{i \in I} A_\ell \rightarrow M$ given by $(a_i)_{i \in I} \mapsto \sum_i a_i m_i$.

Thus M must be a quotient of a semisimple module. The statement follows from 1.18. \square

Definition. Denote by $\mathcal{T}(A)$ the set of isomorphism classes of simple A -modules.

We call a simple module N of type $\tau \in \mathcal{T}(A)$ if the isomorphism class of N is τ .

Given $M \in A\text{-Mod}$, $\tau \in \mathcal{T}(A)$, we set $M_\tau = \sum_{N \subset M \text{ simple of type } \tau} N$ [note that it must be a direct sum of some submodules of type τ .] M_τ is called the τ -isotypic component.

If $M = M_\tau$ then we say that M is isotypic of type τ .

Remark. By 1.12, $\{I \subset A_\ell \mid \text{maximal left ideal}\} \rightarrow \mathcal{T}(A)$ given by $I \mapsto$ isomorphism class of A/I is surjective. Thus $\mathcal{T}(A)$ is a set.

Exercise: This map is bijective (HW2).

Notation. $A\text{-Mod}^{\text{ss}}$ = category of semisimple A -modules.

Corollary 1.20. Let $M \in A\text{-Mod}^{\text{ss}}$. Then,

- i) $M = \bigoplus_{\tau \in \mathcal{T}(A)} M_\tau$.
- ii) $\forall \text{ submodule } M' \subset M : M' = \bigoplus_{\tau \in \mathcal{T}(A)} (M' \cap M_\tau)$.

Proof. i) M semisimple $\implies M = \sum_\tau M_\tau$. Fix τ . 1.18 implies that $M' := M_\tau \cap \left(\sum_{\tau' \neq \tau} M_{\tau'} \right)$ is semisimple.

If $M' \neq 0$ then $\exists N \subset M'$ which is simple. $N \subset M_\tau$ implies N must be of type τ . However, $N \subset \sum_{\tau' \neq \tau} M_{\tau'}$, which implies that N must not be of type τ . This is a contradiction. Therefore, $M' = 0$.

This means the sum is direct, i.e. $M' = \bigoplus_\tau M_\tau$.

ii) 1.18 implies that M' must be semisimple. Therefore $M' = \bigoplus_\tau M'_\tau$.

Clearly $M'_\tau \subset M' \cap M_\tau$.

Conversely, $M' \cap M_\tau$ is semisimple by 1.18. Therefore, $M' \cap M_\tau = \sum_{N \subset M' \cap M_\tau, N \text{ simple}} N$. Note that each N is of type τ . Thus $M' \cap M_\tau = M'_\tau$.

□

Proposition 1.21. If $M, M' \in A\text{-Mod}^{\text{ss}}$, then $\text{Hom}_A(M, M') = \coprod_{\tau \in \mathcal{T}(A)} \text{Hom}_A(M_\tau, M'_\tau)$.

Proof. Suppose $f \in \text{Hom}_A(M, M')$. Consider $f|_{M_\tau} : M_\tau \rightarrow f(M_\tau) \subset M'$.

Note that $f(M_\tau) = \sum_{N \subset M \text{ simple of type } \tau} f(N)$. Thus $f(M_\tau) \subset M'_\tau$.

□

Proposition 1.22. $M \in A\text{-Mod}^{\text{ss}}$. For a submodule $U \subset M$ TFAE:

- i) $\forall f \in \text{End}_A(M), f(U) \subset U$.
- ii) U is a direct sum of some $M_\tau, \tau \in \mathcal{T}(A)$.

Proof. ii \implies i is a direct consequence of 1.21.

i \implies ii: 1.20 and 1.18 imply $U = \bigoplus_\tau U_\tau$.

If $U_\tau \subsetneq M_\tau$, 1.15 $\implies M_\tau = U_\tau \bigoplus_{i \in I} N_i$ where the sum is non-zero and each N_i is of type τ .

Then one finds $f : U \rightarrow M$ such that $f(U) \not\subset U$. Can extend f to a homomorphism $f : M \rightarrow M$.

□

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Corollary 1.23. Let A be semisimple. For a subset $U \subseteq A$ TFAE:

- i) U is an ideal of A . (By convention we mean a two-sided ideal).
- ii) U is a direct sum of isotypic components of A_ℓ .

Proof. i \implies ii: U is an ideal $\implies U$ is a submodule of A_ℓ , and U is stable under ‘right multiplication by a ’. We denote this right multiplication by a map to be ${}_Aa : A_\ell \rightarrow A_\ell$, ${}_Aa(b) = ba$. So U is stable under all ${}_Aa$.

$$1.5 \implies \text{End}_A(A_\ell) = A^o.$$

Therefore, $\forall f \in \text{End}_A(A_\ell) : f(U) \subset U$. Statement ii follows from 1.22.

ii \Leftarrow i: same argument in reverse order. \square

Corollary 1.24. Let A be semisimple.

- i) The isotypic components of A_ℓ are precisely the minimal ideals of A , and every ideal is a direct sum of minimal ideals.
- ii) A has only finitely many minimal ideals.
- iii) $|\mathcal{T}(A)| < \infty$.

Proof. i) $I \subseteq A_\ell$ a direct sum of isotypic components $\stackrel{1.23}{\iff} I$ is an ideal.

Therefore, I is an isotypic component $\iff I$ is a minimal ideal.

- ii) Write $A = \bigoplus_{\tau \in \mathcal{T}(A)} (A_\ell)_\tau$. Then we can write $1 = \sum_\tau a_\tau$ where $a_\tau \in (A_\ell)_\tau$ and all but finitely many a_τ are zero.

Then, $A = \sum_{\tau, a_\tau \neq 0} Aa_\tau \subseteq \bigoplus_{a_\tau \neq 0} (A_\ell)_\tau \subseteq A$.

- iii) 1.18: any simple A -module N is isomorphic to a submodule of A_ℓ . Suppose τ is the type of N . Then, $(A_\ell)_\tau \supseteq N$.

Since there are only finitely many isotypic components, there are only finitely many $\tau \in \mathcal{T}(A)$. \square

Example. 1) Let K_1, \dots, K_n be fields. Then, $A = K_1 \times \dots \times K_n$ is semisimple, and $K_i = \{0\} \times \dots \times \{0\} \times K_i \times \{0\} \times \dots \times \{0\}$ is a simple module, and these exhaust all of $\mathcal{T}(A)$.

- 2) Suppose K is a field, and let $A = M_n(K)$. We can write A_ℓ as a direct sum of columns, i.e. $A_\ell = Ae_1 \oplus \dots \oplus Ae_n$.

Here, $Ae_i \stackrel{A}{\cong} K^n$ is up to isomorphism the only simple A -module. Denote by τ this type. $(A_\ell)_\tau$ is the only isotypic component.

- 3) $A = M_{n_1}(K_1) \times \dots \times M_{n_s}(K_s)$ has s isotypic components.

Lemma 1.25. Let $M \in A\text{-Mod}^{\text{ss}}$ and suppose $M = \bigoplus_{i \in I} N_i$ and $M = \bigoplus_{j \in J} N'_j$ with simple submodules $N_i, N'_j \subset M$.

Then \exists bijection $\sigma : I \rightarrow J$ such that $\forall i \in I : N'_{\sigma(i)} \stackrel{A}{\cong} N_i$. In particular, I and J must have the same cardinality.

Proof. WLOG we may assume $M = M_\tau$. Let N be a fixed simple module of type τ .

Then $N \cong N_i \cong N'_j$ for all $i \in I, j \in J$.

Put $D = \text{End}_A(N)^\circ$. 1.44 \implies D is a division algebra.

Then, the map $D \times \text{Hom}_A(N, M) \rightarrow \text{Hom}_A(N, M)$ given by $(d, f) \mapsto f \circ d$ makes $\text{Hom}_A(N, M)$ into a D -module.

Moreover, $\text{Hom}_A(N, M) = \bigoplus_{i \in I} \text{Hom}_A(N, N_i)$.

However, $\text{Hom}_A(N, N_i) \cong \text{Hom}_A(N, N) = \text{End}_A(N) \cong D^\circ$. Therefore, $\text{Hom}_A(N, N_i)$ is free of rank 1 over D .

Then, $\dim_D \text{Hom}_A(N, M) = |I|$.

By the same argument, $\dim_D \text{Hom}_A(N, M) = |J|$ using the other decomposition. \square

Definition. If we write $M \in A\text{-Mod}^{\text{ss}}$ as $M = \bigoplus_{i \in I} N_i$ where N_i are simple, then $\ell_A(M) = |I|$ is called the *length* of M (as an A -module).

The *length of a semisimple algebra* A is by definition $\ell_A(A_\ell)$.

Given a simple module N of type τ we also use the notation:

$$M : \tau = M : N := \ell_A(M_\tau)$$

Remark. If N is simple and $D = \text{End}_A(N)^\circ$, then $M : N = \dim_D \text{Hom}_A(N, M)$ and $\ell_A(M) = \sum_{\tau \in \mathcal{T}(A)} M : \tau$

Theorem 1.26 (Jacobson Density Theorem). Let $M \in A\text{-Mod}^{\text{ss}}$ and $C = \text{End}_A(M)$ and consider the canonical map:

(1) $A \rightarrow \text{End}_C(M)$, $a \mapsto a_M = [m \mapsto am]$. Note that this map is not necessarily A -linear. But it is C -linear.

Then, the image of (1) is ‘dense’ in $\text{End}_C(M)$ in the following sense:

$\forall f \in \text{End}_C(M)$ and $\forall x_1, \dots, x_n \in M, \exists a \in A$ such that $\forall 1 \leq i \leq n : f(x_i) = a \cdot x_i$ (2).

If M is finitely generated as a C -module, the map (1) is surjective.

Proof. Consider $M' := M^{\oplus n} \in A\text{-Mod}^{\text{ss}}$ and $x := (x_1, \dots, x_n) \in M'$.

1.16 $\implies \exists M'' \subset M'$ with the property $M' = M'' \oplus Ax$.

Then we can define a projection p as follows: $p : M' \rightarrow M'$ as follows: every element of $M' = M'' + Ax$ can be written as $m'' + ax$ where $m'' \in M''$. Then, $p(m'' + ax) := ax \in M'$.

Then, $p \in \text{End}_A(M') =: C'$.

Now define $\tilde{f} : M' \rightarrow M'$ as follows: $\tilde{f}(y_1, \dots, y_n) = (f(y_1), \dots, f(y_n))$.

1.7 $\implies \text{End}_C(M) \xrightarrow{\cong} \text{End}_{C'}(M')$ given by $f \mapsto \tilde{f}$ is a C' -linear bijection.

Thus, $\tilde{f} \circ p = p \circ \tilde{f}$.

Therefore, $(f(x_1), \dots, f(x_n)) = \tilde{f}(x) = \tilde{f}(p(x)) = p(\tilde{f}(x)) = a \cdot x$ for some $a \in A$ [since $\text{im } p = Ax$].

Therefore, $\forall 1 \leq i \leq n : f(x_i) = ax_i$. \square

Corollary 1.27. Let $N \in A\text{-Mod}$ be simple and $D = \text{End}_A(N)$. Consider N as a D -vector space. Let x_1, \dots, x_n be linearly independent over D . Then for any $y_1, \dots, y_n \in N \exists a \in A \forall 1 \leq i \leq n : y_i = ax_i$.

Proof. We apply 1.26 with $C = \text{End}_A(N) = D$.

We extend (x_1, \dots, x_n) to a D -basis of N . Then we can choose $f \in \text{End}_{D=C}(N)$ such that $f(x_i) = y_i$ for all $1 \leq i \leq n$. \square

Tuesday, 1/27/2026

Noetherian and Artinian Modules (E. Artin, 1898-1962, IU 1938-1946)

Definition. A module M is called:

- i) *Noetherian* if every non-empty subset of submodules of M has a maximal element.
- ii) *Artinian* if every non-empty subset of submodules of M has a minimal element.

An algebra A is called *artinian* (resp. *artinian*) if A_ℓ is noetherian (resp. artinian).

Remark 1.28. i) M is noetherian (resp. artinian) iff every increasing (resp. decreasing) (countable) chain is stationary (i.e. all members are the same from some index on).

- ii) For a division ring D and V a D -Mod, TFAE:

$$V \text{ is noetherian} \iff V \text{ is artinian} \iff \dim_D(V) < \infty.$$

- iii) If $M = \bigoplus_{i \in I} M_i$ with all M_i non-zero and $|I| = \infty$, then M is neither noetherian nor artinian.

- iv) \mathbb{Z} is not artinian: $(2) \supsetneq (6) \supsetneq (24) \supsetneq (120) \supsetneq \dots$. \mathbb{Z} is noetherian, as is any PID.

- v) \mathbb{Q}/\mathbb{Z} (as a \mathbb{Z} -module) is not noetherian: $\frac{1}{2}\mathbb{Z}/\mathbb{Z} \subsetneq \frac{1}{6}\mathbb{Z}/\mathbb{Z} \subsetneq \frac{1}{24}\mathbb{Z}/\mathbb{Z} \subsetneq \dots$

$$\mathbb{Q}/\mathbb{Z} \text{ is also not artinian: } M = \sum_{p \text{ prime}} \frac{1}{p}\mathbb{Z}/\mathbb{Z} \supsetneq \sum_{p>2} \frac{1}{p}\mathbb{Z}/\mathbb{Z} \supsetneq \sum_{p>3} \frac{1}{p}\mathbb{Z}/\mathbb{Z} \supsetneq \dots$$

But for $\mathbb{Z}_{(p)}$ [the localization of \mathbb{Z} at p], the module $\mathbb{Q}/\mathbb{Z}_{(p)}$ is artinian.

- vi) If K is a field, then any finite dimensional K -algebra A is noetherian and artinian.

- vii) \exists noetherian and artinian K -algebra A such that A° is neither noetherian nor artinian.

- viii) We'll prove: A artinian $\implies A$ is noetherian.

Proposition 1.29. M is noetherian iff every submodule is finitely generated.

Proof. \implies : given $N \subset M$ set $S = \{N' \subset N \mid N' \text{ is f.g.}\}$.

$0 \in S \implies S \neq \emptyset$, and by assumption S has a maximal element $N_0 \subseteq N$. It is easy to show that $N_0 = N \implies N$ is f.g.

\impliedby : Let $N_1 \subseteq N_2 \subseteq \dots$ be submodules of M . By assumption we can show that $N = \bigcup_i N_i$ is finitely generated. Let m_1, \dots, m_s be generators of N . Then $\exists i_0 : m_1, \dots, m_s \in N_{i_0}$. Thus $N = N_{i_0} = N_{i_0+1} = \dots$. \square

Remark. For M to be noetherian it does not suffice that M is finitely generated.

If A is noetherian and M is finitely generated, then M is noetherian.

Proposition 1.30. Let M be a module and $N \subset M$ be a submodule.

- i) N and M/N are noetherian $\iff M$ is noetherian.
- ii) N and M/N are artinian $\iff M$ is artinian.

Proof. HW3 □

Corollary 1.31. Let p be the property ‘noetherian’ or ‘artinian’.

- i) Suppose $M = \bigoplus_{i=1}^n M_i$. Then M is p iff all M_i are p .
- ii) If an algebra A is p , every $M \in A\text{-Mod}^{\text{f.g.}}$ is p .

Suppose,

$$0 \rightarrow N \hookrightarrow M \rightarrow M/N \rightarrow 0$$

say: M is an extension of M/N by N .

Proof. i) Reduce to the case $n = 2$ and apply 1.30.

- ii) A is $p \implies A_\ell$ is $p \xrightarrow{\text{(i)}} A_\ell^{\oplus n}$ is p . 1.30 $\implies A_\ell^{\oplus n}/L$ is p for any submodule $L \subset A_\ell^{\oplus n}$. Any finitely generated module is of this form which implies the assertion.

□

Proposition 1.32. Let $M \in A\text{-Mod}^{\text{ss}}$. TFAE:

- i) M is finitely generated.
- ii) M is a finite direct sum of simple submodules.
- iii) M is artinian.
- iv) M is noetherian.

Proof. Write $M = \bigoplus_{i \in I} N_i$ where N_i are simple. Assume i, and let m_1, \dots, m_s be generators. $\forall j : 1 \leq j \leq s \exists I_j \subset I$ such that $|I_j| < \infty$ such that $m_j \in \bigoplus_{i \in I_j} N_i$.

Then, $m_1, \dots, m_s \in \bigoplus_{i \in \bigcup_{j=1}^s I_j} N_i = M$ which implies ii.

ii \implies iii, iv is implied by 1.31.

iv \implies i: Noetherian implies finitely generated.

To complete the full circle, assume iii. Use remark 1.28. HW3. □

Corollary 1.33. Every semisimple algebra A is noetherian and artinian.

Proof. A_ℓ is semisimple and f.g., $A_\ell = A \cdot 1_A$. Apply 1.32. □

Thursday, 1/29/2026

Remark. The study of Semisimple Algebras has very little overlap with Commutative Algebras, namely finite product of fields.

There is a generalization: *Azumaya algebras*.

Goro Azumaya: IU (1968-1992)

Proposition 1.34 (Definition of Composition Series). For $M \in A\text{-Mod}$ TFAE:

- i) M is both artinian and noetherian.
- ii) M has a *composition series*: a chain of submodules:

$$M = M_n \supsetneq M_{n-1} \supsetneq \cdots \supsetneq M_0 := 0$$

where M_i/M_{i-1} is simple for $1 \leq i \leq n$ [$n = 0$ if $M = 0$].

n is called the *length of the composition series*.

Proof. ii \implies i: Suppose M has a composition series of some length. Then M_1 must be simple. If there is an M_2 , M_2/M_1 must be simple. We have the exact sequence:

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_2/M_1$$

1.30 then implies that M_2 is noetherian and artinian. The fact that M must be both noetherian and artinian follows from induction.

i \implies ii: HW3. □

Theorem 1.35 (Jordan-Hölder Theorem for Modules). If M has a composition series:

- 1) $M = M_n \supsetneq M_{n-1} \supsetneq \cdots \supsetneq M_0 = 0$.
- 2) $M = L_m \supsetneq L_{m-1} \supsetneq \cdots \supsetneq L_0 = 0$.

then $m = n$. Furthermore, the simple quotient are the same under permutation: $\exists \sigma \in S_n$ such that $\forall 1 \leq i \leq n : L_i/L_{i-1} \xrightarrow{A} M_{\sigma(i)}/M_{\sigma(i)-1}$.

Definition. A module M is called of *finite length* if M satisfies the equivalent conditions of 1.34.

The length of any composition series is called *the length of M* , denoted by $\ell(M)$.

Remark. i) If M is of finite length and N is a submodule, then N is of finite length. Furthermore,

$$\ell(M) = \ell(N) + \ell(M/N).$$

We use the exact sequence:

$$0 \rightarrow N \rightarrow M \rightarrow M/N.$$

ii) M semisimple of finite length, i.e. $M = \bigoplus_{i=1}^n N_i$ where N_i are simple $\implies \ell(M) = n$.

M is not always the direct sum of the simple quotients. For example, suppose K is a field and $A = \{n \times n \text{ upper triangular } K\text{-matrices}\}$ acting on K^n . Then Ke_1 is a submodule but Ke_2 is not one.

$$K^n \supsetneq Ke_1 \oplus \cdots \supsetneq Ke_{n-1} \oplus \cdots \supsetneq Ke_1 \oplus \cdots \oplus Ke_{n-2} \supsetneq \cdots \supsetneq Ke_1 \supsetneq 0.$$

Hence $\ell(K^n) = n$.

Definition. A module N is called *indecomposable* if $N \neq 0$ and N has no direct summand other than 0 and N .

i.e. We cannot write $N = A \oplus B$ where A, B are non-zero submodules. For example, K^n in the previous example is indecomposable as an A -module.

Example. i) Again, suppose K is a field and $A = \{n \times n \text{ upper triangular } K\text{-matrices}\}$ acting on K^n . Then K^n is indecomposable as an A -module.

ii) Any simple module is indecomposable.

iii) Suppose K is a field, and $A = K[x]$ acting on K^n where $x \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & & 0 \\ & \ddots & \ddots & 0 & \\ & & \ddots & 1 & \\ & & & & 0 \end{bmatrix}$ [jordan block]. Then K^n is indecomposable as an A -module.

Proposition 1.36. If M is artinian or noetherian, then it is a direct sum of finitely many indecomposable modules.

Proof. Assume M is artinian. Suppose $M \neq 0$ and M is decomposable. Then, \exists non-zero submodules M_1, M_2 such that $M = M_1 \oplus M_2$.

Consider $S(M) = \{0 \subsetneq N \subsetneq M : N \text{ is a direct summand of } M\}$ $M_1 \in S(M) \implies S(M) \neq \emptyset$.

M is artinian. Therefore, $S(M)$ must have a minimal element N_0 . N_0 must be indecomposable, since any non-trivial decomposition of N_0 contradicts its minimality.

Write $M = N_0 \oplus M'_0$.

Consider $S' = \{M' \subset M : \exists M_0 \subset M : M = M_0 \oplus M' \text{ and } M_0 \text{ is a finite direct sum of indecomposables}\}$.

Then, $M'_0 \in S'$. Thus, $S' \neq \emptyset$. Therefore S' contains a minimal element which we claim is M' .

Suppose $M' \neq 0$. If M' is indecomposable then $M = (M_0 \oplus M') \oplus 0$ which contradicts the minimality of M' , since $0 \in S'$.

Hence M' is indecomposable. Therefore, $S(M') \neq \emptyset$. M' must also be artinian, therefore $S(M')$ has minimal element N'_0 which is indecomposable. We write $M' = N'_0 \oplus M'' \implies M = (M_0 \oplus N'_0) \oplus M''$ where M'' is minimal in S' , which is a contradiction.

Noetherian case: HW3. □

Lemma 1.37. Let $M \in A\text{-Mod}$ and $g \in \text{End}_A(M)$.

i) If g is surjective and M is noetherian $\implies g$ is injective.

ii) If g is injective and M is artinian $\implies g$ is surjective.

Proof. HW3. □

Lemma 1.38. Let M be of finite length and $f \in \text{End}_A(M)$. Then \exists decomposition $M = U \oplus N$ such that $f(U) \subset U, f(N) \subset N$ and $f|_U$ is bijective and $f|_N$ is nilpotent, $(f|_N)^k = 0$.

Proof. HW3. □

Lemma 1.39. Let M be of finite length and indecomposable. Then every endomorphism $f \in C := \text{End}_A(M)$ is either bijective or nilpotent.

Moreover, $I = \{f \in C : f \text{ is nilpotent}\}$ is the unique maximal (2-sided) ideal of C .

Proof. Suppose f is not bijective. 1.39 implies that $M = U \oplus N$ where $f(U) \subset U, f(N) \subset N, f|_U$ is bijective, $f|_N$ is nilpotent. Since f is not bijective, $M \neq U$. Therefore, $N \neq 0$. Since M must be indecomposable, $U = 0$ and $M = N$. Therefore, f must be nilpotent.

Now we prove that I must be an ideal.

$$f \in I, h \in C \implies \ker(h \circ f) \neq 0 \implies h \circ f \in I.$$

$$f \in I, h \in C \implies \text{im}(f \circ h) \subsetneq M \implies f \circ h \text{ is not bijective} \implies f \circ h \in I.$$

We need to show I is closed under addition. Suppose $f, g \in I$ but $f + g \notin I$. Then $f + g$ must be invertible.

$$h(f+g) = 1 \implies hg = 1 - hf. \text{ Since } hf \text{ is nilpotent it follows that } hg \text{ is invertible, which is a contradiction.} \quad \square$$

Theorem 1.40 (Krull-Remak-Schmidt). Let M be a module of finite length. Suppose $M = N_1 \oplus \cdots \oplus N_m = N'_1 \oplus \cdots \oplus N'_n$ with indecomposable submodules N_i, N'_j . Then $m = n$ and $\exists \sigma \in \mathcal{S}_n$ such that $\forall 1 \leq i \leq n : N'_i \cong N_{\sigma(i)}$.

Proof. By induction on $\ell(M)$. If $\ell(M) = 1$ there is nothing to show. Let $\ell(M) > 1$ and suppose the statement is true for all modules of length $< \ell(M)$.

Let $\iota_j : N_j \hookrightarrow M$ be the inclusion and $\pi_j : M \rightarrow N_j$ the projection onto N_j .

Let $p_j = \iota_j \circ \pi_j \in \text{End}_A(M)$. Then p_j is an idempotent, $p_j \circ p_j = p_j$ and $p_j \circ p_i = 0$ if $i \neq j$. Furthermore $p_1 + \cdots + p_m = \text{id}_M$.

We can do the same for the other decomposition: we get p'_j where $p'_1 + \cdots + p'_n = \text{id}_M$ etc.

Consider: $\text{End}_A(N_1) \ni f_j = \pi_1 \circ p'_j \circ \iota_1 : N_1 \hookrightarrow M \xrightarrow{\pi'_j} N'_j \xrightarrow{\iota'_j} M \rightarrow N_1$.

$$f_1 + f_2 + \cdots + f_n = \pi_1 \circ \underbrace{(p'_1 + \cdots + p'_n)}_{\text{id}_M} \circ \iota_1 = \text{id}_{N_1}.$$

1.39 implies that $\exists 1 \leq j \leq n$ such that f_j is bijective. After renumbering, we may assume that f_1 is bijective.

Consider $g = p'_1 \circ p_1 : M \rightarrow M$. Note that $\pi_1 \circ g = f_1 \circ \pi_1$.

Set $h = g + p_2 + \cdots + p_m : M \rightarrow M$. Then $p_1 \circ h = p_1 \circ g = \iota_1 \circ \pi_1 \circ g = \iota_1 \circ f_1 \circ \pi_1$. We want to show that h is bijective.

Suppose $x \in M$ such that $h(x) = 0$.

$\implies 0 = p_1(h(x)) = p_1(g(x)) = \iota_1(f_1(\pi_1(x))) = 0$. Since ι_1 and f_1 are injective, it follows that $\pi_1(x) = 0$. Therefore $x \in N_2 \oplus \cdots \oplus N_m =: M'$. Since $g = p'_1 \circ p_1$ it follows that $g(x) = 0$.

$0 = h(x) = g(x) + p_2(x) + \cdots + p_m(x) = x$ where $g(x) = 0$ and $p_j(x) \in N_j$. Therefore h is injective.

It follows from 1.37 that h is bijective.

Since h is an automorphism and $h|_{N_k} = \text{id}_{N_k}$ for $k \geq 2$ it follows that $M = N'_1 \oplus N_2 \oplus \cdots \oplus N_m$. By quotienting out N'_1 it follows that $N_2 \oplus \cdots \oplus N_m \cong N'_2 \oplus \cdots \oplus N'_n$. The theorem follows from induction. \square

Tuesday, 2/3/2026

2 Wedderburn Theory

Recall: An algebra A is called *simple* if A is non-zero and it has no ideals other than 0 and A .

Recall that by ideal we mean ideals that are both left and right ideals. There can be left ideals that are not right ideals and vice versa. Meaning, an algebra A being simple doesn't necessarily imply that the module A_ℓ is simple.

Proposition 2.1. Let A be simple. TFAE:

- i) A is semisimple.
- ii) A is artinian.
- iii) A possesses a minimal left ideal N .

Minimal ideals are by convention non-zero. This forces A to be non-zero from iii as well.

Recall by saying A is noetherian/artinian we mean A_ℓ is noetherian/artinian.

Proof. i \implies ii: Corollary 1.33 (because A_ℓ is semisimple and finitely generated, hence it has finite length).

ii \implies iii: Trivial.

iii \implies i: Let $0 \neq N \subset A$ be the minimal left ideal. Then, $0 \neq NA = \sum_{a \in A} Na$ is a 2-sided ideal. Since A is simple, $NA = A$.

N minimal left ideal $\implies N$ is simple as an A -module. Na is the image of N under the A -module map $N \rightarrow Na, x \mapsto xa$. If Na is non-zero, it is the non-zero image of a simple module, which implies Na is simple. Then, NA is a sum of simple modules: $A = NA = \sum_{Na \neq 0} Na$ is a sum of simple modules. Therefore, 1.16 $\implies A$ must be semisimple. \square

Corollary 2.2. Let A be simple and semisimple and $N \subset A$ a minimal left ideal. Then, $A_\ell \cong \bigoplus_A N^{\oplus m}$ for some $m > 0$, and A_ℓ is hence isotypic.

Proof. 2.1 implies the existence of a minimal ideal N . Then, $A_\ell = \sum_{a \in A, Na \neq 0} Na$ where Na are simple. By 1.17, $A_\ell \cong \bigoplus_{i \in I} Na_i$ for some subset $\{a_i : i \in I\}$ of A . WLOG we assume that $Na_i \neq 0$ for all $i \in I$.

Then, the identity $1_A = \sum_{j \in J \subset I} n_j a_j$ for some finite $J \subset I$ and $n_j \in N$. Therefore, $\forall a \in A : a = \sum_{j \in J} a n_j a_j$. Note that $a n_j a_j \in Na_j$ since N is a left ideal. Therefore, $A = \bigoplus_{i \in J} Na_i$. Therefore $J = I$. Furthermore, for each $i \in I$, $Na_i \cong N$.

Therefore, $A_\ell \cong \bigoplus_{i \in I} Na_i \cong N^{\oplus |I|}$.

□

Proposition 2.3. Let A be a non-zero semisimple algebra. TFAE:

- i) A is simple.
- ii) A_ℓ is isotypic.
- iii) $|\mathcal{T}(A)| = 1$. Recall that $\mathcal{T}(A)$ is the set of isomorphism classes of simple A -modules.

Proof. i \implies ii follows from 2.2.

ii \implies iii follows from 1.18: every simple module is isomorphic to a submodule of A_ℓ .

Recall 1.24 which says that minimal ideals are precisely the isotypic components of A_ℓ . in fact, $A_\ell = \bigoplus_{\tau \in \mathcal{T}(A)} (A_\ell)_\tau$.

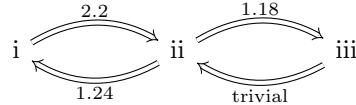
If N is simple of type τ $\implies (A_\ell)_\tau \cong \bigoplus_{i \in I_\tau} N$.

If N is simple $A_\ell/L \cong N \rightarrow N$ appears as a direct summand in $A_\ell = \bigoplus_{\tau \in \mathcal{T}(A)} (A_\ell)_\tau$. $(A_\ell)_\tau \neq 0$ for all $\tau \in \mathcal{T}(A)$.

iii \implies ii is trivial.

ii \implies i follows from 1.24.

Summarizing,



□

Proposition 2.4. If D is a division algebra, then $M_n(D)$ is a simple artinian algebra for any $n \geq 1$.

Proof. Set $V = (D^o)^{\oplus n} \implies A := \text{End}_{D^o}(V) \cong M_n((D^o)^o) = M_n(D)$.

By 1.13: $A_\ell \stackrel{(*)}{=} \underbrace{V \oplus \cdots \oplus V}_{n \text{ copies}}$ and V is simple as A -module (1.13). Then A_ℓ is isotypic and semisimple because of *. Then by 2.3 A is simple. □

Recall: $Z(A) :=$ center of A .

Proposition 2.5. A simple $\implies Z(A)$ is a field.

Proof. Pick non-zero element of the center $a \in Z(A) \setminus \{0\}$. Note that $Aa = aA$ must be a non-zero two-sided ideal, and since A is simple it follows that $aA = A$. $aA = A \implies \exists b \in A$ such that $ab (= ba) = 1$. Therefore a must have a two-sided inverse.

Given $c \in A$: $(cb - bc)a = c(ba) - (ba)c = c1 - 1c = c - c = 0$. Multiplying by b on the right, it follows that $(cb - bc)ab = 0 \implies cb - bc = 0 \implies cb = bc$ for all $c \in A$. Therefore $b \in Z(A)$. □

Theorem 2.6. A non-zero semisimple algebra A has only finitely many distinct minimal ideals A_1, \dots, A_n . Each A_i is a unital algebra in its own right with the induced addition and multiplication from A . Note that the identity in A_i is not equal to the identity in A if $n > 1$.

Moreover, $A = A_1 \times \dots \times A_n$ is the product of the algebras A_1, \dots, A_n (with componentwise addition and multiplication), and each A_i is simple and artinian.

Conversely, if A_1, \dots, A_n are simple artinian algebras, then $A := A_1 \times \dots \times A_n$ is semisimple and A_1, \dots, A_n are precisely the minimal ideals of A .

Note that direct sum and product carry the same meaning in this case. We generally use direct sums when we're thinking about the object as a module, and products when we're thinking about the object as a ring.

Proof. $1.24 \implies A = A_1 \oplus \dots \oplus A_n$ where A_1, \dots, A_n are the isotypic components of A_ℓ . These are precisely the (2-sided) ideals.

We also have: $A_i A_j \subset A_1 \cap A_j = 0$ for $i \neq j$.

Write $1_A = e_1 + \dots + e_n$ with $e_i \in A_i$. $1_A = 1_A \cdot 1_A \implies 1_A = e_1^2 + \dots + e_n^2$. Taking projections it follows that $\forall 1 \leq i \leq n : e_i^2 = e_i$.

$\forall a \in A_i, a = a \cdot 1_A = ae_1 + \dots + ae_i + \dots + ae_n = ae_i = e_i a$.

Therefore, e_i is the identity element in A_i .

If $0 \neq I \subset A_i$ is an ideal, then I is an ideal in A . However, since A_i is minimal, it follows that $I = A_i$. Therefore A_i must be simple.

If $0 \neq I \subset A_i$ is a left ideal, then I is a left ideal in A . Since A is a semisimple algebra A must be artinian. Thus A_i must also be artinian.

The converse is easy to check (HW4) □

Corollary 2.7. Let $A = A_1 \times \dots \times A_n$ be a semisimple algebra with simple algebras A_1, \dots, A_n . Then,

$$Z(A) = Z(A_1) \times \dots \times Z(A_n)$$

is a product of fields, by 2.5. In particular, $Z(A)$ is a field if and only if A is simple.

Corollary 2.8. A commutative semisimple algebra A is a product of finitely many fields: $A = K_1 \times \dots \times K_n$.

These fields are uniquely determined as subsets of A . Namely, they are the minimal ideals of A .

Remark 2.9. A commutative artinian algebra A is semisimple if and only if its nilradical $\{a \in A \mid \exists n > 0 : a^n = 0\}$ is zero.

\implies follows from 2.8.

\Leftarrow uses the theory of radical at the end of chapter 28. Direct proof on HW4.

Corollary 2.10. Let K be a field. A finite dimensional commutative K -algebra having no non-zero nilpotent elements is a product of finitely many field extensions K_i/K with $[K_i : K] < \infty$.

Proof. By the preceding remark, A is semisimple. By 2.8, it is a product of fields K_1, \dots, K_n , all of which have finite degree over K . □

Now we state a key theorem of this course.

Theorem 2.11 (Wedderburn's Theorem). An artinian algebra A is simple if and only if it is isomorphic (as rings) to a matrix algebra $M_n(D)$ for a division algebra D and $n > 0$:

$$A \cong M_n(D)$$

D and n are uniquely determined by A .

Proof. By the remark, A is semisimple. By 2.8 it is a product of fields K_1, \dots, K_n , all of which have finite degree $/K$. \square

$$A \text{ semisimple} \implies A \cong M_{n_1}(D_1) \times \dots \times M_{n_r}(D_r).$$

$$[D_1] \cdot [D_2] = [D_1 \otimes_K D_2] = [M_n(D_3)] = [D_3].$$

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Remark. Prop 2.4: $M_n(D)$ is simple.

Notation. If K is a (skew) field then $[V : K] := \dim_K(V)$.

Theorem 2.12. Let A be a simple artinian algebra, and let N be a simple A -module (unique upto isomorphism). Set $D = \text{End}_A(N)$ which is a division algebra.

i) The canonical map $A \rightarrow \text{End}_D(N), a \mapsto a_N, a_N(x) = ax$, is an isomorphism, and $r := \dim_D(N)$ is finite and equal to $\ell_A(A_\ell)$.

Therefore, $A \cong M_r(D^\circ)$. $Z(A)$ is a field, isomorphic to $Z(D)$.

ii) If M is a non-zero f.g. A -module, then $B = \text{End}_A(M)$ is simple and artinian, and $B \cong M_n(D)$, with D as in (i) and $n = \ell_A(M)$. If K is a field and A is a K -algebra, then,

$$[M : K]^2 = [A : K][B : K]$$

and all these dimensions are finite provided one of them is finite.

Proof. Step 1: 2.1 implies A is semisimple. 2.3 $\implies |\mathcal{T}(A)| = 1$. Let N be a simple A -module, M a f.g. A -module. 1.19 $\implies M$ is semisimple. $\stackrel{1.32}{\implies} M \cong N^{\oplus n}$. $n = \ell_A(M)$.

$$1.7 \implies B := \text{End}_A(M) \cong \text{End}_A(N^n) \cong M_n(\text{End}_A(N)) \stackrel{\text{def}}{=} M_n(D).$$

Thus B is simple and artinian.

Step 2: Apply Step 1 with $M = A_\ell \cong N^{\oplus m}$, $m = \ell_A(A_\ell)$.

$$\text{Step 1} \implies \text{End}_A(A_\ell \cong M_m(\text{End}_A(N))) = M_m(D).$$

Proposition 1.5 (HW1): $\text{End}_A(A_\ell) = A^\circ$.

$$\implies A \cong \text{End}_A(A_\ell)^\circ \cong M_m(D)^\circ \stackrel{HW1}{\cong} M_m(D^\circ).$$

Step 3: It remains to show: $\boxed{m(= \ell_A(A_\ell)) = \dim_D(N) =: r.}$

Fix an isomorphism of A -modules $\varphi : A_\ell \xrightarrow{\cong} N^{\oplus m}$.

Consider the diagram:

$$\begin{array}{ccc}
 A & \xrightarrow{a \mapsto a_N} & \text{End}_D(N) \\
 1.5 \implies \cong \downarrow \alpha & & 1.7 \implies \cong \downarrow \beta \\
 \text{End}_{A^o}(A_\ell^o) = \text{End}_{\text{End}_A(A_\ell)}(A_\ell^o) & \xrightarrow[\varphi^*]{\cong} & \text{End}_{\text{End}_A(N^m)}(\underbrace{N^m}_{=N' \text{ in 1.7}}) \\
 & &
 \end{array}$$

$$\varphi^*(\tau)(\underbrace{x}_{\in N' = N^m}) = \varphi(\tau(\varphi^{-1}(x))).$$

Easy to check: this diagram is commutative.

Upshot: the canonical map $A \rightarrow \text{End}_D(N)$ is an isomorphism!

$\implies \text{End}_D(N)$ is artinian.

$\implies \text{End}_D(N)$ is artinian $\underset{\text{easy (HW4)}}{\implies} [N : D] = \dim_D(N) < \infty$.

$\implies N \underset{D}{\cong} D^{\oplus r}$.

1.13 $\implies \text{End}_D(N)_\ell \underset{\text{as } \text{End}_D(N)\text{-modules}}{\cong} N^r$.

In the other hand, $\text{End}_D(N)_\ell \cong M_n(D^o) \cong \text{End}_D(D)$.

Therefore, $r = \dim_D(N) = \ell_A(A_\ell) = m$.

Consider again $A \xrightarrow{\cong} \text{End}_D(N) \cong M_r(D^o)$.

$\implies Z(A) \xrightarrow{\cong} Z(M_r(D^o)) \underset{\text{easy (HW4)}}{=} Z(D^o) = Z(D)$.

Step 4: Let $M \in A\text{-Mod}^{\text{fg}}$. Assume A is a K -algebra. Write $M = N^n$.

Then, $[M : K] = n[N : K] = n[N : D][D : K] = nr[D : K]$.

i $\implies [A : K] = r^2[D : K]$.

Step 1 $\implies [B : K] = n^2[D : K]$.

This shows the formula. □

Theorem 2.13. Suppose D_1, D_2 are division algebras and $M_r(D_1) \cong M_s(D_2)$ as rings. Then $r = s$ and $D_1 \cong D_2$.

Proof. Step 1: Suppose N, N' are A -modules over A (any algebras). Then any isomorphism $f: N \rightarrow N'$ of A -modules induces an isomorphism of rings $\text{End}_A(N) \rightarrow \text{End}_A(N')$ given by $\tau \mapsto f \circ \tau \circ f^{-1}$.

Step 2: If $g: A \rightarrow A'$ is an isomorphism of simple artinian algebras, and $N \in A\text{-Mod}$ and $N' \in A'\text{-Mod}$ are simple, then $a \cdot x' =: g(a) \cdot x'$ gives N' the structure of an A -module. Since g is an isomorphism, N' becomes a simple A -module. 2.3 $\implies N' \cong N$ as A -modules.

Step 1 $\implies \text{End}_A(N) \cong \text{End}_A(N') = \text{End}_{A'}(N')$ immediately.

Step 3: D division algebra $\xrightarrow[1.13]{\text{HW1}} D^r$ is a simple module over $A := \text{End}_D(D^r) (\cong M_n(D^o))$ and $\text{End}_{M_r(D^o)}(D^r) \xrightarrow[1.8]{\text{HW1}} D$.

Now suppose $M_r(D_1) \cong M_s(D_2)$.

$$\text{HW1} \implies M_r(D_1^o) \xrightarrow[\text{HW1}]{\cong} M_n(D_1)^o \cong M_s(D_2)^o \xrightarrow[\text{HW1}]{\cong} M_s(D_2^o).$$

$N := D_1^r$ is a simple $M_r(D_1^o)$ -module.

$N' := D_2^s$ is a simple $M_s(D_2^o)$ -module.

$$\text{Step 2} \implies D_1 \xrightarrow[\text{(*)}]{\cong} \text{End}_{M_r(D_1^o)}(N) \cong \text{End}_{M_s(D_2^o)}(N') \xrightarrow[\text{(*)}]{\cong} D_2.$$

Step 4: If $A := M_r(D) \xrightarrow[\text{rings}]{\cong} M_s(D) =: B$ then $r = \ell_A(A_\ell) = \ell_B(B_\ell) = s$. \square

Proof of Wedderburn's Theorem. Combine 2.12 and 2.13. \square

Corollary 2.14. i) A simple algebra A is artinian if and only if A^o is artinian, in which case $\ell_A(A_\ell) = \ell_{A^o}(A_\ell^o)$.

ii) A is semisimple if and only if A^o is semisimple, in which case $\ell_A(A_\ell) = \ell_{A^o}(A_\ell^o)$.

Proof. i) A simple artinian $\xrightarrow[2.13]{\cong} A \cong M_n(D) \implies A^o \cong M_n(D^o)$.

ii) follows from 2.6 and (i). \square

Corollary 2.15. A unital associative algebra A is semisimple if and only if $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ where D_1, \dots, D_r are division algebras and n_1, \dots, n_r and the isomorphism classes of D_r, \dots, D_r are uniquely determined upto permutation.