M634 Algebraic Varities 2

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HW every week. Office Hours: Monday 2:30 - 3:30 RH 251

Textbook: Ravi Vakil

Recalling some important background material.

1) Let A be a ring [commutative with identity by default]. Then $\dim A = \sup\{n \mid \exists p_0 \subsetneq \cdots \subsetneq p_n \subset A\}$ max chain of prime ideals.
2) Naively one might assume every noetherian ring has a finite dimension. This is not true, there are noetheriann ring with ∞ dimension.
3) Let $(A, m, k), k = A/m$ be a local noetherian ring. Then $\dim A \leq \dim_k m/m^2$. In particular, $\dim A < \infty$.
4) Given a prime ideal $p \subset A$, the height $\text{ht}(p) = \text{codim}(p) = \sup\{n \mid \exists p_0 \subsetneq \cdots \subsetneq p_n = p\}$.

We can ask the following quantities: $\dim A$ [consider all prime ideal chains], $\dim A/p$ [consider all prime ideal chains [strictly] containing p] and $\text{ht}(p)$ [consider all prime ideal chains ending with p].

A natural question: $\dim A \stackrel{?}{=} \dim A/p + \text{ht}\, p$?

Answer: Sadly, not in general, even for noetherian ring. Check 12.3.13.

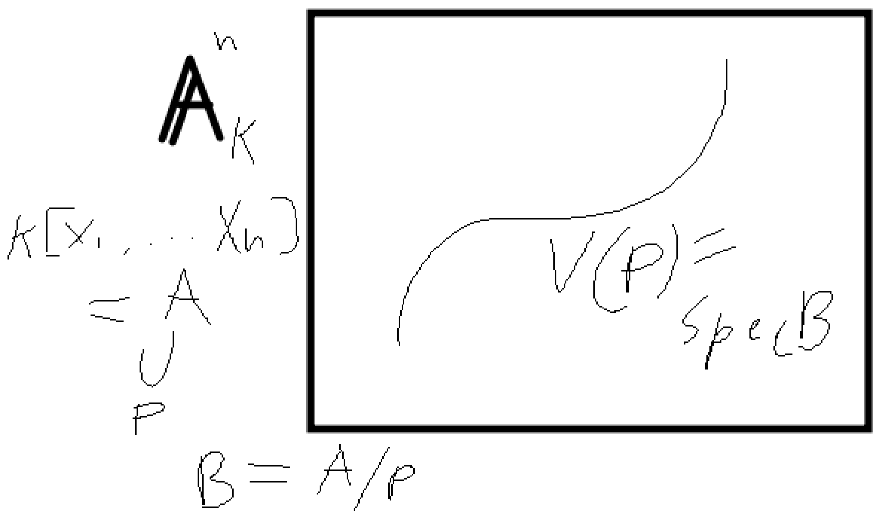
$A \supset p$ so that p is maximal, then height 1. $\dim A = 2$.

Remark: answer is yes if A is a *catenary ring.*

Definition. A is a *catenary ring* if $\forall p \subset q \subset A$, 'every strictly increasing chain of prime ideals from p to q' is contained in such a chain with maximal length.

We start with a field k, suppose A/k is finitely generated as a k-algebra.

Exercise 12.2.H : show that if A is a localization of a finitely generated algebra over a field, then A is catenary.



n = dim A = dim B + ht(P)

Theorem 1 (Prime Ideal Theorem of Krull [PIT).]

Let A be a noetherian ring. Let $f \in A$.

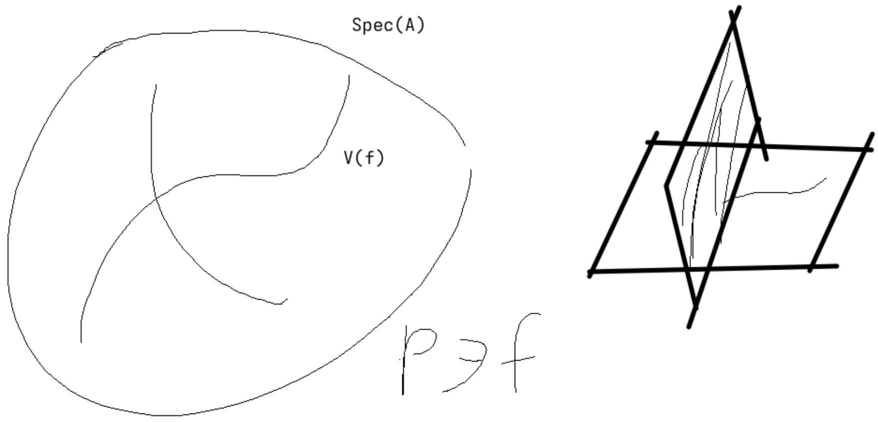


Figure 1: Geometric depiction

Then every minimal prime $p \ni f$ has height ≤ 1.

If further f is not a zero divisor [which geometricallly means zero set of f doesn't cover a whole connected component of A], then every minimal $p \ni f, \operatorname{ht}(p) = 1$.

Let us state an auxillary result.

Theorem 2 (12.3.10 Krull's Height Theorem)**.** Let $X = \operatorname{Spec} A$ where A is noetherian.

Suppose Z is an irreducible component of $V(f_1, \cdots, f_r) \subset X$. Then $\operatorname{codim} Z \leq r$.

Lemma 3 (12.3.9)**.** Let A be notherian, $p_0 \subsetneq \cdots \subsetneq p_n$ a strict chain in A. Suppose $f \in p_n \setminus p_0$.

Then $\exists$ strict chain $p_0 = q_0 \subsetneq q_1[\ni f] \subsetneq \cdots \subsetneq q_n = p_n$.

Note that in the catenary case this is trivial.

Corollary 4. Let (A, m) be a local noetherian ring. $f \in m$.

Then $\dim A/(f) \geq \dim A - 1$.

Proof. Choose a maximal chain in A.

$$p_0 \subsetneq \cdots \subsetneq m \text{ in } A$$

If $f \in P_0$ then $\overline{p_0} \subsetneq \cdots \subsetneq \overline{m}$ is still a strict chain. Thus $\dim A/(f) = \dim A$.

If $f \notin p_0$ then $f \in p_n \setminus p_0$.

By lemma, $\exists$

$$p_0 = q_0 \subsetneq q_1[\ni f] \subsetneq \cdots \subsetneq q_n = p_n$$

Quotient implies $\overline{q}_1 \subsetneq \cdots \subsetneq \overline{q}_n = \overline{m}$ is a valid chain.

So $\dim A/(f) \geq n - 1$.

□

Exercise 12.2.M : Suppose X is a locally finite type k-scheme of pure dimension $= n$. Take any field extension of the base field K/k. Then, the *extended scheme* $X_k = K \times_k X$ also has pure dimension $= n$.

If A/k is a f.g. domain then $\dim A = \operatorname{trdeg} K(A)/k$.

Lemma 5 (Nakayama's Lemma)**.** V1: Let M be a finitely generated A-module. $I \subset A$ an ideal. $\varphi \in \operatorname{End}_A M$ such that $\varphi(M) \subset IM$.

Then $\exists a_1, \cdots, a_n \in I$ such that,

$$\varphi^n + a_1\varphi^{n-1} + \cdots + a_n = 0 \text{ in } \langle A, \varphi \rangle \subset \operatorname{End}_A(M)$$

Even though $\operatorname{End}_A M$ is not commutative, $\langle A, \varphi \rangle$ is.

Interesting even if $I = A$.

V2: Let A, m be a local ring and M a f.g. A-module so that $mM = M$. Then $M = 0$.

Note that finitely generated is important: $\mathbb{C}[x]_{(x)}$ is a local ring with maximal ideal (x). Suppose $M = \mathbb{C}(x)$ which is not finitely generated. Then $xM = M$ but $M \neq 0$.

V3: Let (A, m, k) be a noetherian local ring. M a f.g. A-module.

Then M/mM is a k-module.

Let $x_1, \cdots, x_n \in M$ be elements of M so that their projections span M/mM. Then they span M.

A domain A is normal (integrally closed) if for any element $x \in K(A)$ such that it is integral over the ring i.e. $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ then $x \in A$.

If A is factorial then A is normal.

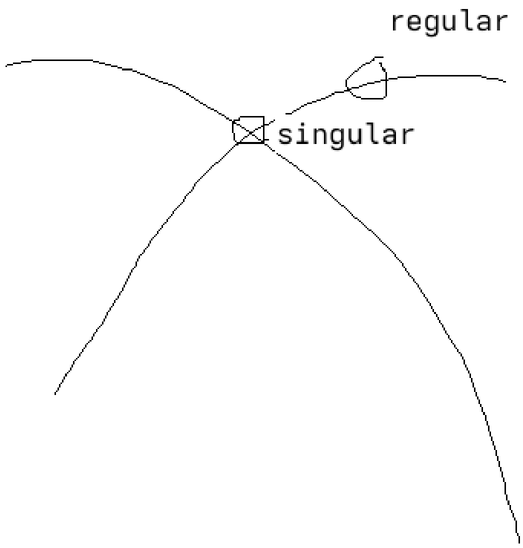
If A is normal then every localization A_P is also normal [5.4.A].

Now we move on to today's topic.

13 Regularity and Smoothness

Given $p \subset A, [p] \in \operatorname{Spec} A$,

Question: when is $\operatorname{Spec} A$ 'manifold like' near $[p]$?



Definition. Let (A, m, k) be local. Then its *Zariski Cotangent Space* is defined to be $T^\vee := m/m^2/k$

It's dual space $\operatorname{Hom}_k(T^\vee, k)$ is the *Zariski Tangent Space.*

Suppose X is a scheme, $p \in X$ a point.

Then the Zariski Cotangnet Space $T^\vee_{X,p}$ is the Zariski Cotangent Space at $\mathcal{O}_{X,p}$

Definition. A noetherian local ring (A, m, k) is regular if $\dim_k m/m^2 = \dim A$. Note that we always have $\geq$, for regularity we need equality.

Definition. A noetherian ring A is regular if A_p is regular for all $p \subset A$.

Example: Let $X = \operatorname{Spec} K[x, y]/(xy) \hookrightarrow \mathbb{A}^2_k$. $A = K[x, y]/(xy)$. We want to take a point and see if it is regular.

A is not regular in the origin. Consider $(\overline{x}, \overline{y}) \subset K[x, y]/(xy)$. $A_{(0,0)} \supset m = (\overline{x}, \overline{y})$, $m/m^2 = k \cdot \overline{x} + k \cdot \overline{y}$ so $(\overline{x}, \overline{y}) = m/m^2$.

Dimension of the space is 2, ring has dimension 1.