Talks

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1 Ext Duality in derived category $\mathcal{O}$ and its variants

Let F be any field of char 0. Classically $\mathbb{C}$.

$\underline{G}$ = split reductive algebraic group. e.g. $\underline{G} = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$ etc.

Suppose $\underline{G} \supset \underline{P}$, where $\underline{P}$ is the parabolic subgroup, a group consisting of upper triangular submatrices, e.g. $\begin{bmatrix} GL_{n_1} & & * \\ 0 & GL_{n_2} & \\ 0 & 0 & GL_{n_3} \end{bmatrix}$

$\underline{B} \supset \underline{P}$ where $\underline{B}$ is the Borel subgroup.

$\underline{T} \supset \underline{B}$ where $\underline{T}$ is the maximal split torus. e.g. $\underline{T} \cong \mathbb{G}_m^d$.

We take the successive lie algebras to get $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{b} \supset \mathfrak{t}t$.

$\mathfrak{t}^* = \mathrm{Hom}_F(\mathfrak{f}, F)$ weight space.

$U = U(\mathfrak{g})$ the universal enveloping algebra. U-mod is the category of U-left modiules.

1.1 Category $\mathcal{O}$ and its variants

$M \in U$-mod, M is called <u>$\mathfrak{t}$-split</u> if it is a direct sum of generalized weight spaces i.e. $M = \bigoplus_{\lambda \in \mathfrak{t}^*} M_\lambda^\infty$.

$M_\lambda^\infty = \bigcup_{i>0} M_\lambda^i$.

$M_\lambda^i = \{m \in M \mid \forall x \in \mathfrak{t} : (x - \lambda(x))^i \cdot m = 0\}$ generalized weight space.

$\Pi(M) = \{\lambda \in \mathfrak{t}^* \mid M_\lambda^\infty \neq 0\}$ is the set of weights of $M \supset \Pi(M)_{alg}$ = algebraic weights $\ni \lambda$ is algebraic if $\exists$ algebraic char $\underline{T} \xrightarrow{\chi} \mathbb{G}_m$,

$d_\chi = \lambda \colon \mathfrak{t} = \mathrm{Lie}(\underline{T}) \to \underline{Lie}(\mathbb{G}_m) = F$.

Definition. $M \in U$-mod belongs to $\mathcal{O}^{p,\infty}$ if:

1) M is f.g. as U-module.
2) M is $\mathfrak{t}$-split
3) $\mathfrak{p}$ acts <u>locally finite</u>, i.e. $\forall m \in M, \dim_F(U(\mathfrak{p}) \cdot m) < \infty$.

If 4) $\Pi(M) = \Pi(M)_{alg}$ then $M \in \mathcal{O}^{\mathfrak{p},\infty}_{alg}$.

If 5) $M = \bigoplus_{\lambda \in \Pi(M)} M^i_\lambda$ then $M \in \mathcal{O}^{\mathfrak{p},i}_{alg}$

Classically (meaning in the work of Bernstein-Gelfand-Gelfand, which is why it is often called the BGG category $\mathcal{O}$): $\mathfrak{g}$ is a semisimple complex lie algebra. Then $\mathfrak{p} = \mathfrak{b}$, the borel subalgebra. Also, the i in the definition of generalized eigenspace is always 1, meaning we just have regular eigenspaces.

Note that, all finite dimensional modules trivially satisfy this definition.

There are $\mathfrak{sl}_2$ modules satisfying the definition that are not even weight modules!

We consider the examples:

1) Finite dimensional modules: $\mathcal{O}^{\mathfrak{g}}$ [here $\mathfrak{p} = \mathfrak{g}$] $\subseteq \mathcal{O}^{\mathfrak{p},\infty} \subseteq \mathcal{O}^{\mathfrak{b},\infty} = \mathcal{O}^\infty$.

2) Verma modules: consider the linear form $\mathfrak{t} \xrightarrow[\text{weight}]{\lambda} F = F_\lambda$. We have $\mathfrak{b} \twoheadrightarrow \mathfrak{t}$ and $\mathfrak{b} \to F$. We can consider F_λ as a $\mathfrak{b}$-module, i.e a $U(\mathfrak{b})$-module $\rightsquigarrow M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} F_\lambda \in \mathcal{O}^{\mathfrak{b},1} \subset \mathcal{O}^{\mathfrak{b},i}$. This is the so-called universal highest weight module. There are analogues of highest weight modules for others. Note that this is an abelian category. Quotients of Verma modules is allowed.

   For example, $\mathfrak{g} = \mathfrak{sl}_2, \lambda_n \left(h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = n \in \mathbb{Z}_{\geq 0} \rightsquigarrow 0 \to M(\lambda_{-n-2}) \hookrightarrow M(\lambda_n) \twoheadrightarrow \underbrace{V_n}_{\text{irrep of } \dim n+1} \to 0.$

   Remark: Category $\mathcal{O} = \mathcal{O}^{\mathfrak{b},1}$ is not closed under extensions in U-mod.

   e.g. $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{t} \to \mathfrak{gl}(\mathbb{F}^{\oplus 2}), \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$0 \to M(\lambda_0) \to [\mathcal{O}^{\mathfrak{b},2} \in, \mathcal{O}^{\mathfrak{b},1} \notin] U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} F^{\oplus 2} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} F_0 = M(\lambda_0) \to 0$$

1.2 Motivation

Let $F/\mathbb{Q}_p$ be a finite extension, $G = \underline{G}(F)$. e.g. $G = \mathrm{GL}_n(\mathbb{Q}_p)$.

We would like to have a commutative diagram:

$$\begin{array}{ccc} D^{\mathfrak{b}}_{\mathcal{C}_G}(\mathcal{M}_G) & \xrightarrow{\mathbb{D}_G} & D^{\mathfrak{b}}_{\mathcal{C}_G}(\mathcal{M}_G) \\ \mathcal{F}^G_P \uparrow & & \mathcal{F}^G_P \uparrow \\ D^{\mathfrak{b}}(\mathcal{O}^{\mathfrak{p},\infty})_{\text{alg}} & \xrightarrow{\mathbb{D}_{\mathfrak{g}}} & D^{\mathfrak{b}}(\mathcal{O}^{\mathfrak{p},\infty}_{\text{alg}}) \end{array}$$

Here, $\mathcal{M}_G = D(G)$-mod, where $D(G)$ is the locally analytic distribution algebra.

We have $\mathbb{Q}_p[G] \overset{\text{dense}}{\hookrightarrow} D(G), U(\mathfrak{g}) \hookrightarrow D(G)$.

$\mathcal{M}_G \supset \mathcal{C}_G =$ coadmissible modules (finiteness condition).

$\underbrace{D^{\mathfrak{b}}_{\mathcal{C}_G}(\mathcal{M}_G)}_{\text{triangulated}} = \{ M^\bullet \in D^{\mathfrak{b}}(\mathcal{M}_G) \mid H^\bullet(M^\bullet) \text{ is coadmissible} \}$.

Theorem 1 (Schneider-Teilelbamn 2005)**.** $\mathbb{D}_G(M^\bullet) = \mathrm{RHom}_{D(G)}(M; D_c(G))$.

$D_c(G) =$ dual space of locally analytic functions with compact support. Dualizing objects.

$\mathbb{D}_G \circ \mathbb{D}_G = \mathrm{id}$. It's a right module $\rightsquigarrow$ left module using $g \mapsto g^{-1}$.

Functors $\mathcal{F}_P^G : \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p},\infty} \to \mathcal{C}_G \subset \mathcal{M}_G$, 'globalization functors' '$\mathfrak{g}$-reps $\rightsquigarrow$ G-reps'.

Theorem 2 (Orlib-S (2015), Agarwal-S (2021))**.** $\mathcal{F}_P^G$ is exact.

For example, if we take a Verma module $M(\lambda)$ where $\lambda = d_\chi$ [a derivative of an algebraic character],

$$\mathcal{F}_B^G(M(\lambda)) = \left(\mathrm{Ind}_B^G(\chi^{-1})^{\text{loc. an.}}\right)'$$

The $'$ here denotes a topological dual space.

If $M \in \mathcal{O}^{\mathfrak{p},1}$ is simple and $M \notin \mathcal{O}^{\mathfrak{g},1}$, $\mathfrak{g} \supset \mathfrak{p}$ then $\mathcal{F}_P^G(M)$ is a topologically simple module.

1.3 First approach to find the duality $\mathbb{D}_\mathfrak{g}$

Theorem 3. $U(\mathfrak{g})$ is a noetherian ring of finite local dimnesion equal to $\dim((\mathfrak{g}))$.

$U = U(\mathfrak{g})$ is therefore a dualizing object itself. i.e. the functor $\mathrm{RHom}_U(-, U)$ is an involutive anti-isomorphism of $D^\mathfrak{b}(U\text{-mod})$.

Fact: $\forall M \in \mathcal{O}^{\mathfrak{p},\infty} : \forall q \geq 0 : \mathrm{Ext}_U^q(M, U) \in \mathcal{O}^{\mathfrak{p},i}$

Question: If $M \in \mathcal{O}^{\mathfrak{p},\infty}$: is $\mathrm{RHom}_U(M, U)$ naturally quasi-isomorphic to a complex of modules in $\mathcal{O}^{\mathfrak{p},\infty}$?

Answer: Yes, if $\mathfrak{p} = \mathfrak{b}$.

Theorem 4 (Coulembier - Mazorchuk)**.** The category $\mathcal{O}^{\mathfrak{b},\infty}$ is extension-full in U-mod.

Extension-full means, for all objects $M, N \in \mathcal{O}^{\mathfrak{b},\infty}$ and $\forall q$: the Yoneda Ext group $\text{Y-Ext}^q_{\mathcal{O}^{b,\infty}}(M, N) \xrightarrow{\cong} \mathrm{Ext}^q_{U\text{-mod}}(M, U)$.

We need Yoneda Ext Group since the category doesn't have enough projective and injective objects.

Corollary 5. The natural functor $D^\mathfrak{b}(\mathcal{O}^{\mathfrak{b},\infty}) \to D^\mathfrak{b}_{\mathcal{O}^{\mathfrak{b},\infty}}(U\text{-mod})$ is an equivalence of categories. The subscript $\mathcal{O}^{\mathfrak{b},\infty}$ means the cohomology is in $\mathcal{O}^{\mathfrak{b},\infty}$.

$\mathbb{D}$ acts on $D^\mathfrak{b}(\mathcal{O}^{\mathfrak{b},\infty})$ by transport of structure.

$\mathbb{D}$ acts on $D^\mathfrak{b}_{\mathcal{O}^{\mathfrak{b},\infty}}$.

1.4 Second approach

First approach doesn't generalize to general parabolic subalgebras $\mathfrak{p}$. Take $\mathfrak{p} = \mathfrak{g}$ semisimple $= \mathcal{O}^{\mathfrak{g},\infty} = \mathcal{O}^{\mathfrak{g},1}$ which is the category of all f.g. g-modules.

$\implies \mathrm{Ext}^q_{\mathcal{O}^\mathfrak{g}}(M, N) = 0$ for all $q > 0$.

Even for $\mathfrak{sl}_2 : H^3(\mathfrak{g}, M) \neq 0$.

Note $H^3(\mathfrak{g}, M) = \mathrm{Ext}^3_\mathfrak{g}(\mathbf{1}, M)$.

Solution in general:

$M \mapsto \mathrm{RHom}_U(M, U)$

$M \mapsto \mathrm{Hom}_U(M, U) = 0$

$M \mapsto \mathrm{Ext}^e_U(M, U) = E^{\mathfrak{p}}(M)$ where $e = \dim \mathfrak{p}$.

$$\mathrm{Ext}_U^{\dim(\mathfrak{p})}\left(U \otimes_{U(\mathfrak{p})} \underbrace{W}_{\text{f.d.}}, U\right) \cong (W^* \otimes \wedge^{\text{top}} \mathfrak{p}^*) \otimes_{U(\mathfrak{p})} U$$

Theorem 6. $\mathrm{RHom}_U(M, U)$ [here $M \in \mathcal{O}^{\mathfrak{p},\infty}$] equals $\mathrm{RE}^{\mathfrak{p}}(M)[-e]^{\mathcal{O}^{\mathfrak{b},\infty}}$

Superscript denotes the cohomology is in $\mathcal{O}^{\mathfrak{b},\infty}$.